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MATHEMATICS

On Surface Waves and Tidal Waves near a Promontory.

BY

SUDHANSUKUMAR BANERJEE.

[*Read January 26th, 1919.*]

The only problem of the free vibrations of a rotating sheet of gravitating liquid of small uniform depth which may be considered to have been completely solved is the one in which the boundary is circular.[1] When the boundary is not circular, the difficulty of a complete solution is much greater. Lord Rayleigh[2] obtained a partial solution for the case when the boundary is rectangular, applicable when the angular velocity of rotation is small. In a recent paper Proudman[3] has used Lord Rayleigh's approximate theory of diffraction to solve a number of other problems, namely, those of the diffraction of a plane tidal wave by an elliptic island, by a semi-elliptic cape, by a rectangular bay and by a passage between one sea and another.

In the present paper the theory of multiform solution as developed by Sommerfeld has been used to solve the problem of the diffraction of surface waves by a long promontory which for simplicity has been assumed to be either a semi-infinite plane bounded by a straight edge or a wedge forming a definite angle. The nature of the tidal waves on flat rotating sheet of water near a promontory of the above mentioned shapes has also been determined. A method has also been given for determining the free tidal oscillations in a rotating circular sector, a problem[4] which Proudman found to be exceedingly difficult to solve.

[1] Kelvin, *Phil Mag.*, Aug. 1880; Lamb, *Hydrodynamics*, §§ 208, 209, 210.

[2] Lord Rayleigh, *Phil. Mag.*, V, pp. 297-301 (1903) [*Scientific Papers*, vol. V, p. 93]. See also *Proc. Roy. Soc.*, A. Vol. 82, p. 448.

[3] Proudman, *Proc. Lond. Math. Soc.*, Vol. 14, p. 89 (1915).

[4] *Proc. Lond. Math. Soc.*, Vol. 12, p. 453, (1913).

Case I. *Surface Waves—Diffraction by a semi-infinite plane barrier.*

We shall first consider the case of waves propagated in a liquid of uniform depth under the action of gravity.

Let the plane of the undisturbed surface be the plane of xy, and let the axis of y be measured in the direction of propagation of the wave and the axis of z vertically upwards. Let the barrier occupy the half of the xz plane for which x is positive.

Using cylindrical coordinates, we see that since the motion is supposed to be irrotational, the velocity potential Φ must satisfy the equation

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}+\frac{\partial^2\Phi}{\partial z^2}=0. \quad \ldots \quad \ldots \quad (1)$$

At the bottom of the liquid, when $z=-h$,

$$\frac{\partial\Phi}{\partial z}=0. \quad \ldots \quad \ldots \quad (2)$$

On the surface of the barrier, we have

$$\frac{\partial\Phi}{\partial\theta}=0, \text{ when } \theta=0 \text{ and } \theta=2\pi. \quad \ldots \quad \ldots \quad (3)$$

On the free surface the condition to be satisfied is

$$\frac{\partial^2\Phi}{\partial t^2}+g\frac{\partial\Phi}{\partial z}=0, \text{ when } z=0. \quad \ldots \quad \ldots \quad (4)$$

Let us now assume that the incident wave is given by the velocity potential

$$\Phi_0=\text{A}\cosh k(z+h)\cos k(x\sin\theta_1+y\cos\theta_1)\cos pt,$$

where

$$p^2=gk\tanh kh, \quad \ldots \quad \ldots \quad (5)$$

and A is a constant.

It is easy to see that this expression satisfies (1), (2) and (4) and therefore represents a set of plane waves which can be maintained on the surface of an infinite liquid of constant depth h. This set of waves will be incident on the barrier at an angle θ_1. If now Φ represents the velocity potential of the total disturbance when the barrier is present in the liquid, we can assume the following expression for Φ

$$\Phi=\text{A}\cosh k(z+h)f(r,\theta)\cos pt. \quad \ldots \quad \ldots \quad (6)$$

It is now easy to see by substitution in the differential equation (1), that $f(r, \theta)$ must satisfy the equation

$$\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}+k^2 f=0 \quad \ldots \qquad \ldots \quad (7)$$

and also since on the surface of the barrier $\frac{\partial f}{\partial \theta}=0$ when $\theta=0$ and $\theta=2\pi$, $f(r, \theta)$ must have the same form as the expression obtained by Sommerfeld for the diffraction of plane polarised light by a perfectly reflecting semi-infinite screen, the magnetic force in the incident light being parallel to the edge of the screen. Hence $f(r, \theta)$ must be given by the expression

$$f(r, \theta)=\tfrac{1}{2}\cos k(x \sin\theta_1+y\cos\theta_1)+\tfrac{1}{2}\cos k(x\sin\theta_1-y\cos\theta_1)$$

$$+\tfrac{1}{2}\sqrt{\frac{k}{\pi}}\int^{r+x\sin\theta_1+y\cos\theta_1}\cos\left\{\frac{\pi}{4}+k(x\sin\theta_1+y\cos\theta_1-u)\right\}\frac{du}{\sqrt{u}}$$

$$+\tfrac{1}{2}\sqrt{\frac{k}{\pi}}\int^{r+x\sin\theta_1-y\cos\theta_1}\cos\left\{\frac{\pi}{4}+k(x\sin\theta_1-y\cos\theta_1-u)\right\}\frac{du}{\sqrt{u}}. \quad \ldots \quad (8)$$

This by means of a result recently obtained by Hargreaves[1] can be written in the form

$$f(r, \theta)=\tfrac{1}{2}\cos k(x\sin\theta_1+y\cos\theta_1)+\tfrac{1}{2}\cos k(x\sin\theta_1-y\cos\theta_1)$$

$$+\left[J_{\frac{1}{2}}(kr)\cos\frac{\theta}{2}\left(\cos\frac{\theta_1}{2}+\sin\frac{\theta_1}{2}\right)+J_{\frac{5}{2}}(kr)\cos\frac{5\theta}{2}\left(\cos\frac{5\theta_1}{2}+\sin\frac{5\theta_1}{2}\right)+\ldots\right]$$

$$+\left[J_{\frac{3}{2}}(kr)\cos\frac{3\theta}{2}\left(\cos\frac{3\theta_1}{2}-\sin\frac{3\theta_1}{2}\right)+J_{\frac{7}{2}}(kr)\cos\frac{7\theta}{2}\left(\cos\frac{7\theta_1}{2}-\sin\frac{7\theta_1}{2}\right)+\ldots\right]. \quad \ldots \quad (9)$$

[1] Hargreaves, "A diffraction problem and an asymptotic theorem in Bessel's series," *Phil. Mag.*, Aug., 1918.

For normal incidence $f(r, \theta)$ can be written in the form

$$f(r, \theta) = \cos ky + J_{\frac{1}{2}}(kr) \cos \frac{\theta}{2} + J_{\frac{3}{2}}(kr) \cos \frac{3\theta}{2} + J_{\frac{5}{2}}(kr) \cos \frac{5\theta}{2}$$

$$+ J_{\frac{7}{2}}(kr) \cos \frac{7\theta}{2} + \quad \ldots \quad (10)$$

Hence the solution for normal incidence can be written in the form

$$\Phi = A \cosh k(z+h) \cos ky \cos pt$$

$$+ A \cosh k(z+h) \left[J_{\frac{1}{2}}(kr) \cos \frac{\theta}{2} + J_{\frac{3}{2}}(kr) \cos \frac{3\theta}{2} + \ldots \right] \cos pt \ldots \quad (11)$$

This expression can be used to plot the stream lines graphically on any plane z=constant. The method is to plot first the equipotential lines Φ=const. starting from equidistant points on the positive side of the x-axis. It will be noticed that the equipotential lines curve round the edge of the barrier and then proceed to infinity asymptotically to the negative direction of the x-axis. The stream lines which consist of the orthogonal set of lines are easily drawn and are found, over a considerable portion of the region, to proceed very nearly from the edge of barrier. This is somewhat analogous to the radiation of light from the edge of the screen in the corresponding optical problem.

Case II. *Surface Waves—Diffraction by a wedge-shaped barrier.*

Let the edge of the wedge be chosen as the axis of z, and let r, θ, z be cylindrical coordinates of a point so that the faces of the wedge are given by $\theta=0$ and $\theta=\alpha$ and the space occupied by it is that between $\theta=\alpha$ and $\theta=2\pi$.

If now the incident wave be represented by the real part of the expression

$$\Phi_0 = A \cosh k(z+h) e^{ik[r\cos(\theta-\theta_0)+Vt]} \quad \ldots \quad (12)$$

where $$V^2 = g/k \tanh kh.$$

then it is easy to see that the velocity potential of the total disturbance is given by the real part of the expression

$$\Phi=\frac{A}{2\pi\iota}\cosh k(z+h)e^{ikVt}\int e^{ikr\cos\zeta}\frac{d}{d\zeta}\log(\omega\omega_1)d\zeta.$$

where $\omega=\cos\frac{\pi\zeta}{\alpha}-\cos\frac{\pi}{\alpha}(\theta-\theta_0)$, $\omega_1=\cos\frac{\pi\zeta}{\alpha}-\cos\frac{\pi}{\alpha}(\theta+\theta_0)$, ... (13)

the path of integration being a complete contour which starts from $\infty\iota+\lambda$ and goes to $\infty\iota+\lambda$ without crossing the real axis. The expression for Φ can also be written in the form

$$\Phi=\frac{2\pi A}{\alpha}\cosh k(z+h)\left[\sum_1^\infty J_{\frac{n\pi}{\alpha}}(kr)\frac{\pi}{\sin\frac{n\pi^2}{\alpha}}\cos\frac{n\pi\theta}{\alpha}\sin\frac{n\pi\theta_1}{\alpha}\right]\cos kVt \qquad (14)$$

As in the previous case this expression may be used to plot the steam lines.[1]

Case III. (1) *Tidal Waves near a long narrow Promontory.*

Suppose the sheet of water to be rotating with uniform angular velocity about an axis perpendicular to its plane and let the depth of water (as rotating in free relative equilibrium) be uniform and equal to h. Let ζ denote the elevation of the free surface at any time. Then for a disturbance in which the time t only enters through the factor $e^{\iota\sigma t}$, we have the equation

$$\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}+k^2\zeta=0, \qquad \ldots (15)$$

there being no disturbing force and

$$k^2=\frac{\sigma^2-4\omega^2}{gh}.$$

The boundary condition is given by

$$\iota\sigma\frac{\partial\zeta}{\partial n}+2\omega\frac{\partial\zeta}{\partial s}=0 \qquad \ldots (16)$$

[1] See Wiegrefe, "Über einige mehrwertige Lösungen der Wellengleichung $\nabla^2u+k^2u=0$ und ihre Anwendung in der Beugungstheorie," *Ann. d. Phys.*, Bd. 39, p. 449 (1912).

where $\frac{\partial}{\partial n}$ denotes differentiation along the outward drawn normal to the boundary and $\frac{\partial}{\partial s}$ along the positive direction of the arc. We exclude the cases when $\sigma=0$ and $\sigma^2=4\omega^2$.

For a narrow promontory, the boundary condition reduces to

$$\frac{\iota\sigma}{r}\frac{\partial\zeta}{\partial\theta} + 2\omega\frac{\partial\zeta}{\partial r}=0, \quad \ldots \quad \ldots \quad (17)$$

to be satisfied when $\theta=0$ and $\theta=2\pi$.

Let $$\zeta=\zeta_1+\zeta_2,$$

where ζ_1, ζ_2 are subject to the conditions

$$\frac{\partial\zeta_2}{\partial\theta}=0 \quad \text{and} \quad \frac{\partial\zeta_1}{\partial r}=0, \quad \ldots \quad \ldots \quad (18)$$

when $\theta=0$ and $\theta=2\pi$.

When $\sigma^2>4\omega^2$, we can assume the following expressions for ζ_1 and ζ_2:

$$\zeta_1=\sum_{n=0}^{\infty} A_n J_{2n+1}(kr)e^{\iota\left[\sigma t+\frac{(2n+1)\theta}{2}\right]},$$

$$\zeta_2=\sum_{n=0} \frac{2n+1}{2} J_{2n+1}(kr)e^{\iota\left[\sigma t+\frac{(2n+1)\theta}{2}\right]}.$$

It is obvious from the conditions (17) and (18) that for ζ_2, we should take the real part of the expression assumed and for ζ_1 the imaginary part and *vice-versa*.

The unknown constant A_n is determined from the boundary condition (17) which gives

$$-\sum\frac{\sigma(2n+1)}{2}A_n\frac{J_{\frac{2n+1}{2}}(kr)}{kr}+\omega\sum(2n+1)\frac{dJ_{\frac{2n+1}{2}}(kr)}{d(kr)}=0 \quad \ldots \quad (19)$$

to be satisfied for all values of r.

But since

$$\frac{2n+1}{z} J_{\frac{2n+1}{2}}(z) = J_{\frac{2n-1}{2}}(z) + J_{\frac{2n+3}{2}}(z),$$

$$\frac{d}{dz} J_{\frac{2n+1}{2}}(z) = \tfrac{1}{2}\left\{ J_{\frac{2n-1}{2}}(z) - J_{\frac{2n+3}{2}}(z) \right\}, \quad \ldots \quad (20)$$

the above equation can be written in the form

$$-\frac{\sigma}{2}\sum A_n \left[J_{\frac{2n-1}{2}}(kr) + J_{\frac{2n+3}{2}}(kr)\right]$$

$$+\omega\sum (2n+1)\left[J_{\frac{2n-1}{2}}(kr) - J_{\frac{2n+3}{2}}(kr)\right] = 0.$$

Hence equating the co-efficient of $J_{\frac{2n+1}{2}}(kr)$ to zero, we obtain

$$A_{n-1} + A_{n+1} = \frac{8\omega}{\sigma} \quad \ldots \quad (21)$$

from which we deduce that

$$A_0 = A_1 = A_2 = \ldots = \frac{4\omega}{\sigma}. \quad \ldots \quad \ldots \quad (22)$$

Hence the expression for ζ can be written in the form

$$\zeta = \sum J_{\frac{2n+1}{2}}(kr)\left[\frac{4\omega}{\sigma}\sin\frac{(2n+1)\theta}{2} + \frac{2n+1}{2}\cos\frac{(2n+1)\theta}{2}\right] e^{i\sigma t} \quad (23)$$

This result will only be applicable over limited portions of the region considered, since the fundamental equations are really only valid for limited regions.

(2) *Free Tidal oscillations in a rotating circular sector bounded by* $r=a$, $\theta=0$, $\theta=2\pi$.

To obtain the complete solution for the *free* tidal oscillations in a circular sector bounded by $r=a$, $\theta=0$ and $\theta=2\pi$, we notice from the above expression for ζ and from the boundary condition to be satisfied on the circular rim that in the nth mode of vibration k must be taken to be a root of the equation

$$(2n+1)\omega\, J_{\frac{2n+1}{2}}(ka) + ka\sigma\frac{d}{d(ka)} J_{\frac{2n+1}{2}}(ka) = 0 \quad \ldots \quad (24)$$

Hence the tidal height is given by

$$\zeta = \sum_k B_k \left[\sum_{n=0}^{\infty} J_{\frac{2n+1}{2}}(kr) \left\{ \frac{4\omega}{\sigma} \sin \frac{(2n+1)\theta}{2} \right.\right.$$

$$\left.\left. + \frac{2n+1}{2} \cos \frac{(2n+1)\theta}{2} \right\} \right] e^{i\sigma_k t},$$

where the summation extends over all the roots of the equation (24) and the constants B_k's are determined by the equations

$$\sum_k B_k \left[(2s+1)\omega \, J_{\frac{2s+1}{2}}(ka) + ka\sigma \frac{d}{d(ka)} J_{\frac{2s+1}{2}}(ka) \right] = 0$$

$$[s=0, 1, 2 \ldots n-1, n+1, n+2 \ldots]$$

and the periods of the oscillations in the nth mode are given by k which are the roots of the equation (24).

When $\sigma^2 < 4\omega^2$, we have to replace $J_{\frac{2n+1}{2}}(kr)$ by $I_{\frac{2n+1}{2}}(kr)$.

Case IV. (1) *Tidal Waves near a wedge-shaped Promontory.*

The method given in the previous case can be extended to determine the tidal heights near a wedge-shaped promontory and also the free tidal oscillations in any rotating circular sector.

The boundary condition is

$$\frac{i\sigma}{r} \frac{\partial \zeta}{\partial \theta} + 2\omega \frac{\partial \zeta}{\partial r} = 0$$

to be satisfied when $\theta=0$ and $\theta=a$.

As before we take

$$\zeta = \zeta_1 + \zeta_2, \quad \ldots \quad \ldots \quad (25)$$

where ζ_1, ζ_2 are such that

$$\frac{\partial \zeta_2}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial \zeta_1}{\partial r} = 0$$

when $\theta=0$ and $\theta=a$.

We can assume the following expressions for ζ_1, ζ_2

$$\zeta_1=\sum_{n=1}^{\infty} A_n J_{\frac{n\pi}{a}}(kr)e^{\iota\left(\sigma t+\frac{n\pi\theta}{a}\right)} \qquad \ldots \quad (26)$$

$$\zeta_2=\sum_{n=1}\frac{n\pi}{a} J_{\frac{n\pi}{a}}(kr)e^{\iota\left(\sigma t+\frac{n\pi\theta}{a}\right)}, \qquad \ldots \quad (27)$$

with this restriction that in the final expression we take for ζ_1 the imaginary part and for ζ_2 the real part of the expressions assumed and *vice versa.*

The constant A_n can be determined as in the previous case from the boundary condition

$$\sum\frac{\sigma n\pi}{a}A_n\frac{J_{\frac{n\pi}{a}}(kr)}{kr}+2\omega\sum\frac{n\pi}{a}\frac{d}{d(kr)}J_{\frac{n\pi}{a}}(kr)=0, \qquad \ldots \quad (28)$$

which has to be satisfied for all values of r, by the use of the following expansions[1]

$$\frac{J_{\frac{n\pi}{a}}(kr)}{kr}=a_{n-1}\left[J_{\frac{(n-1)\pi}{a}}(kr)+J_{\frac{(n+1)\pi}{a}}(kr)\right]$$

$$+a_{n-2}\left[J_{\frac{(n-2)\pi}{a}}(kr)+J_{\frac{(n+2)\pi}{a}}(kr)\right]+\text{etc.,} \ \ldots \quad (29)$$

and $$\frac{d}{d(kr)}J_{\frac{n\pi}{a}}(kr)=b_{n-1}\left[J_{\frac{(n-1)\pi}{a}}(kr)-J_{\frac{(n+1)\pi}{a}}(kr)\right]$$

$$+b_{n-2}\left[J_{\frac{(n-2)\pi}{a}}(kr)-J_{\frac{(n+2)\pi}{a}}(kr)\right]+\text{etc.} \ldots \quad (30)$$

$$[2\pi>a\geq\pi]$$

[1] These expansions do not appear to have been given by any previous writer. A proof of these expansions will be given in a subsequent paper.

(2) *Free Tidal oscillations in a rotating circular sector bounded by*

$r=a$, $\theta=0$, and $\theta=\alpha$.

As in the previous case the method can also be used to determine the free tidal oscillations in a circular sector bounded by $r=a$, $\theta=0$, $\theta=\alpha$.

We have

$$\zeta=\sum_k \mathrm{B}_k \left[\sum_{n=1}^{\infty} \mathrm{J}_{\frac{n\pi}{\alpha}}(kr)\left\{\mathrm{A}_n \sin\frac{n\pi\theta}{\alpha}+\frac{n\pi}{\alpha}\cos\frac{n\pi\theta}{\alpha}\right\}\right] e^{i\sigma_k t},$$

where B_k's are determined by the equations

$$\sum_k \mathrm{B}_k \left[\frac{2\omega s\pi}{\alpha}\,\frac{\mathrm{J}_{\frac{s\pi}{\alpha}}(ka)}{a}+\sigma ka\frac{d}{d(ka)}\mathrm{J}_{\frac{s\pi}{\alpha}}(ka)\right]=0$$

$[s=1, 2, \ldots n-1, n+1, n+2 \ldots]$

and the summation extends over all the values of k which are the roots of the equation—

$$\frac{2\omega n\pi}{\alpha}\mathrm{J}_{\frac{n\pi}{\alpha}}(ka)\Big/ka+\sigma\frac{d}{d(ka)}\mathrm{J}_{\frac{n\pi}{\alpha}}(ka)=0.$$

Hence the periods of the free tidal oscillations in a circular sector bounded by $r=a$, $\theta=0$, $\theta=\alpha$ in the nth mode are given by k which are the roots of the equation

$$\frac{2\omega n\pi}{\alpha}\,\frac{\mathrm{J}_{\frac{n\pi}{\alpha}}(ka)}{ka}+\sigma\frac{d}{d(ka)}\mathrm{J}_{\frac{n\pi}{\alpha}}(ka)=0.$$

On the potentials of uniform and heterogeneous elliptic cylinders at an external point.

BY

NIKHILRANJAN SEN.

[*Read February 10th, 1918.*]

1.

The potential of an infinite elliptic cylinder at an external point is generally expressed in the form of an integral and it is well-known that a transformation in conjugate functions would allow the same integral to be represented by a much simpler expression.[1] It is here proposed to express the potential in trigonometrical series. The method followed is that of integration which will be shown to be applicable also in the case of heterogeneity. It will be found that the potential is always expressible as

$$A_0 \log r - \sum_{n=1}^{\infty} A_n \frac{\cos n\theta}{r^n},$$

where A_n in its most general form can be expressed by hypergeometric functions in e^2 (eccentricity), reducing in two special cases to finite binomial forms. This happens when the cylinder is homogeneous or when the density (supposed constant along lines parallel to the axis) at any point on the elliptic section varies inversely as the focal distance of the point. We shall simplify our problem by considering only the logarithmic potential of the elliptic section to which the (Newtonian) potential of the elliptic cylinder is equivalent but for an infinite constant and the constant multiplier 2.

2.

Before proceeding with the solution of the problem proposed above it would be useful to consider the expansion of $(1+e\cos\phi)^{-n}$, $e<1$, in cosines of multiples of ϕ. Expanding $(1+e\cos\phi)^{-n}$ by the binomial

[1] Lamb, Mess. of Math. 1878.

theorem and replacing the powers of $\cos\phi$ by cosines of multiples of ϕ it may be shown that[1]

$$(1+e\cos\phi)^{-n} = \sum_{m=0}^{\infty} A_m^n \cos m\phi$$

where

$$A_m^n = (-1)^m \frac{(n+m-1)!}{(n-1)!\,m!} \frac{e^m}{2^{m-1}} F\left(\frac{n+m}{2}, \frac{n+m+1}{2}, m+1, e^2\right),$$

$$(e<1)$$

where F is the hypergeometric function of the four elements within the parenthesis. Since A_m^n is a Fourier's co-efficient, we have

$$\int_{-\pi}^{\pi} \frac{\cos m\phi\, d\phi}{(1+e\cos\phi)^n} = \pi A_m^n$$

$$= (-1)^m \frac{(n+m-1)!}{(n-1)!\,m!} \frac{\pi e^m}{2^{m-1}} F\left(\frac{n+m}{2}, \frac{n+m+1}{2}, m+1, e^2\right)$$

and

$$\int_{-\pi}^{\pi} \frac{\cos n\phi\, d\phi}{(1+e\cos\phi)^{n+2}} = (-1)^n \frac{(2n+1)!}{(n+1)!\,n!} \frac{\pi e^n}{2^{n-1}} F\left(n+1, \frac{2n+3}{2}, n+1, e^2\right)$$

$$= (-1)^n \frac{(2n+1)!}{(n+1)!\,n!} \frac{\pi e^n}{2^{n-1}} \frac{1}{(1-e^2)^{n+\frac{3}{2}}};$$

also

$$\int_{-\pi}^{\pi} \frac{\cos n\phi\, d\phi}{(1+e\cos\phi)^{n+1}} = (-1)^n \frac{(2n)!}{n!\,n!} \frac{\pi e^n}{2^{n-1}}. \quad F\left(n+\tfrac{1}{2}, n+1, n+1; e^2\right)$$

$$= (-1)^n \frac{(2n)!}{n!\,n!} \frac{\pi e^n}{2^{n-1}} \frac{1}{(1-e^2)^{n+\frac{1}{2}}},$$

n being an integer and $e<1$.

[1] This expansion in another form is given by Gauss.

3.

Taking the focus S as origin let the equation of the ellipse be given by

$$\rho = \frac{l}{1+e\cos\phi}.$$

Let P be any point (ρ, ϕ) within the area of the ellipse and A another point (r, θ) at a sufficient distance from it. Taking the area to be of unit density the potential

$$V = \int_{-\pi}^{\pi} \int_{0}^{\frac{l}{1+e\cos\phi}} \log AP \, \rho d\rho d\phi.$$

Now

$$AP^2 = \rho^2 - 2\rho r \cos(\phi-\theta) + r^2$$

and

$$\log AP = \log r + \tfrac{1}{2} \log \left[1-2\left(\frac{\rho}{r}\right)\cos(\phi-\theta) + \left(\frac{\rho}{r}\right)^2\right]$$

$$= \log r - \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\rho}{r}\right)^n \cos n(\phi-\theta),$$

r being supposed to be greater than the maximum radius vector, *i.e.*, the length of the major axis from S to the remoter vertex of the ellipse.

Hence

$$V = \frac{l^2}{2}\int_{-\pi}^{\pi} \frac{d\phi}{(1+e\cos\phi)^2} \log r - \sum_{n=1}^{\infty} \frac{l^{n+2}}{n(n+2)r^n} \int_{-\pi}^{\pi} \frac{\cos n(\phi-\theta) d\phi}{(1+e\cos\phi)^{n+2}}.$$

But

$$\int_{-\pi}^{\pi} \frac{\cos n\phi \, d\phi}{(1+e\cos\phi)^{n+2}} = (-1)^n \frac{(2n+1)!}{(n+1)!\, n!} \frac{\pi e^n}{2^{n+1}} \frac{1}{(1-e^2)^{n+\frac{3}{2}}}$$

and

$$\int \frac{\sin n\phi \, d\phi}{(1+e\cos\phi)^{n+2}} = 0.$$

Hence

$$\frac{(1-e^2)^{\frac{3}{2}}}{\pi l^2}\,V=\log r-\sum_{n=1}^{\infty}(-1)^n\frac{(2n+1)!}{n.n!\,(n+2)!}\frac{1}{2^{n-1}}\left(\frac{le}{1-e^2}\right)^n\frac{\cos n\theta}{r^n}.$$

But

$$\frac{le}{1-e^2}=ae=CS=c,$$

where C is the centre.

So that we have finally

$$\frac{(1-e^2)^{\frac{3}{2}}}{\pi l^2}\,V=\log r-\sum_{n=1}^{\infty}(-1)^n\frac{(2n+1)!}{n.n!\,(n+2)!}\frac{1}{2^{n-1}}\left(\frac{c}{r}\right)^n\cos n\theta$$

and 2V is the potential of the elliptic cylinder neglecting an infinite constant.

4.

Let us suppose the cylinder to be heterogeneous and any line parallel to the axis to be a line of equal density. Let the density at the point (x, y) on the elliptic section be $f(x, y)$ where f is a rational algebraic integral function in x and y. Such a function is also expressible in a series in ρ and ϕ of which the typical terms are $\rho^p\cos q\phi$ and $\rho^p\sin q\phi$. It will be sufficient for us to work out the case of these two densities.

(i) Suppose the density $\sigma=\rho^p\cos q\,\phi$.

Then as before

$$V=\iint\sigma\log AP\,\rho d\rho d\phi,$$

the integration is to be carried over the entire area of the ellipse. Proceeding exactly as in the previous case we have

$$V=\frac{l^{p+2}}{(p+2)}\int^{\pi}\frac{\cos q\,\phi\,d\phi}{(l+e\cos\phi)^{p+2}}\log r-\sum_{n=1}^{\infty}\frac{l^{n+p+2}}{n(n+p+2)r^n}\int_{-\pi}^{\pi}\frac{\cos q\phi\cos n(\phi-\theta)d\phi}{(1+e\cos\phi)^{n+p+2}}$$

$$=\frac{l^{p+2}}{(p+2)}\int^{\pi}\frac{\cos q\phi\,d\phi}{(1+e\cos\phi)^{p+2}}\log r-\sum_{n=1}^{\infty}\frac{l^{n+p+2}\cos n\theta}{2^n(n+p+2)r^n}.$$

$$\int\frac{[\cos(n+q)\phi+\cos(n-q)\phi]\,d\phi}{(1+e\cos\phi)^{n+p+2}}$$

...egrals from § 2

$$\frac{V}{\pi l^{p+2}} = \frac{A_q^{p+2}}{(p+2)} \log r - \sum_{n=1}^{\infty} \frac{A_{n+q}^{n+p+2} + A_{n-q}^{n+p+2}}{2n(n+p+2)} \left(\frac{l}{r}\right)^n \cos n\theta,$$

where $n-q$ is the positive value of the difference between these two integers.

When $q=p$ one half of the series is expressible in a simpler form; since

$$A_{n+p}^{n+p+2} = (-1)^{n+p} \frac{(2n+2p+1)!}{(n+p+2)!\,(n+p)!} \frac{e^{n+p}}{2^{n+p-1}} \frac{1}{(1-e^2)^{n+p+\frac{3}{2}}}$$

this part of the potential function can be written as

$$\frac{(-1)^p e^p}{2^p (1-e^2)^{p+\frac{3}{2}}} \left[\frac{2\cdot(2p+1)!}{p!(p+2)!} \log r - \sum_{n=1}^{\infty} (-1)^n \frac{(2n+2p+1)!}{n\cdot(n+p+2)!\,(n+p)!} \left(\frac{e}{2r}\right)^n \cos n\theta\right]$$

the other part being expressed in hypergeometric functions. This is possible only within the limits in which such separation of terms is legitimate. A similar simplification is possible when $q=p+1$.

When $q=0$ and $\sigma=\rho^p$ the potential is given by

$$\frac{V}{\pi l^{p+2}} = \frac{A_0^{p+2}}{(p+2)} \log r - \sum_{n=1}^{\infty} \frac{A_n^{n+p+2}}{n(n+p+2)} \left(\frac{l}{r}\right)^n \cos n\theta.$$

It may be noted that when p is a negative integer this formula is applicable with a slight modification. The terms beginning from the first up to the nth where $n=-p+1$ would have their co-efficients in finite forms which it is easy to calculate.

A very interesting case of the above arises when $p=-1$ or the heterogeneity is of such a nature that the density at any point on the elliptic section varies inversely as the focal distance. Since

$$A_n^{n+1} = (-1)^n \frac{(2n)!}{n!\,n!} \frac{e^n}{2^{n-1}} F(n+\tfrac{1}{2}, n+1, n+1; e^2)$$

$$= (-1)^n \frac{(2n)!}{n!\,n!} \frac{e^n}{2^{n-1}} \frac{1}{(1-e^2)^{n+\frac{1}{2}}};$$

we have in such a case

$$\frac{V}{\pi l} = A^1_0 \log r - \sum_{n=1}^{\infty} \frac{A^{n+1}_n}{n(n+1)} \left(\frac{l}{r}\right)^n \cos n\theta$$

and making the above substitutions the potential function is found to be given by

$$\frac{V(1-e^2)^{\frac{1}{2}}}{\pi l} = 2\log r - \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{2^{n-1} n.n!(n+1)!} \left(\frac{e}{r}\right)^n \cos n\theta.$$

(ii) Suppose that the density $\sigma = \rho^p \sin q\phi$.

Then as before

$$V = \frac{l^{p+2}}{(p+2)} \int_{-\pi}^{\pi} \frac{\sin q\phi \, d\phi}{(1+e\cos\theta)^{p+2}} \log r - \sum_{n=1}^{\infty} \frac{l^{n+p+2}}{n(n+p+2)} \int^{\pi} \frac{\sin q\phi \cos n(\phi-\theta) d\phi}{(1+e\cos\phi)^{n+p+2}}.$$

Now
$$\int^{\pi} \frac{\sin q\phi \, d\phi}{(1+e\cos\phi)^{p+2}} = 0$$

and replacing the product in the numerator of the other integral by the sum of two sines we have

$$\frac{V}{\pi l^{p+2}} = - \sum_{n=1}^{\infty} \frac{A^{n+p+2}_{n-q} - A^{n+p+2}_{n+q}}{2n(n+p+2)} \left(\frac{l}{r}\right)^n \sin n\theta.$$

The logarithmic term is absent. The line $\theta=0$, $\theta=\pi$ is a line of zero potential, as is obvious. Also at a great distance from the origin where $\frac{1}{r^2}$, $\frac{1}{r^3}$ etc. can be neglected the potential is approximately given by

$$V = -\frac{\pi l^{p+2}}{2(p+3)} \left[A^{p+3}_{q-1} - A^{p+3}_{q+1}\right] \frac{\sin\theta}{r};$$

hence the corresponding equipotential lines are arcs of circles touching the major axis at the focus.

We can determine all the cases in which the hypergeometric functions appearing as co-efficients in the trigonometrical series are expressible in finite forms. Two cases we have already studied where

they reduce into binomials; let us enquire if in any other cases such reduction is possible. The function $F(\alpha, \beta; \gamma, e^2)$ will be a binomial expansion if either $\gamma=\alpha$ or $\gamma=\beta$.* Taking the most general case (*i*) § 4 we seek to satisfy either of these conditions in both the functions

A^{n+p+2}_{n+q} and A^{n+p+2}_{n-q} by giving suitable values to p and q.

$$A^{n+p+2}_{n+q} = C \times F\left(n+1+\frac{p+q}{2}, n+\frac{3}{2}+\frac{p+q}{2}, n+q+1; e^2\right),$$

hence for the required condition we should have

$$n+q+1=\text{either } n+1+\frac{p+q}{2}$$

or

$$n+\frac{3}{2}+\frac{p+q}{2},$$

i.e.,

$$\left.\begin{array}{l} p=q \\ p+1=q \end{array}\right\}$$

or

and $A^{n+p+2}_{n-q} = C' \times F\left(n+1+\frac{p-q}{2}, n+\frac{3}{2}+\frac{p-q}{2}, n-q+1; e^2\right)$

which in a similar manner gives

$$\left.\begin{array}{l} p=3q \\ p+1=3q \end{array}\right\}.$$

So we are to find p and q such that any of the four following sets of equations should be consistent

$$\left.\begin{array}{l} p=q \\ p=3q \end{array}\right\} \quad \left.\begin{array}{l} p=q \\ p+1=3q \end{array}\right\} \quad \left.\begin{array}{l} p+1=q \\ p=3q \end{array}\right\} \quad \left.\begin{array}{l} p+1=q \\ p+1=3q \end{array}\right\}.$$

q being zero or an integer.

From these four equations we get only two possible solutions namely

$$\left.\begin{array}{l} p=o \\ q=o \end{array}\right\} \quad \begin{array}{l} p=-1 \\ q=o \end{array}$$

* The other complicated forms, *e.g.*, $F\left\{\frac{1}{2}+\frac{1}{2n}, \frac{1}{2n}, 1+\frac{1}{n}; x^2\right\}$ are at seen to be inapplicable here.

answering to the case of homogeneity and to that of the density varying as the inverse focal distance. These are the only two cases in which the potential function for an infinite elliptic cylinder for the outside space is expressible in a trigonometrical series with binomial co-efficients.

6.

In § 3 we have obtained a trigonometrical series for the potential function V for the outside space by integrating log r throughout the entire area of the section. But in course of our analysis in order to make the expansion of the logarithim of the distance PA possible we had to introduce a certain limitation, namely that r should always be greater than ρ, this immediately marks out a circular area with centre S and radius equal to the maximum radius vector within which the point A must not lie. It will now be shown that the series V has a much wider area of convergence which extends even into the limiting circle and consequently from considerations of continuity it represents the potential function everywhere inside that extended area.

It is well-known that the series $\sum(-1)^n\ a_n \cos n\theta$ is convergent if $a_n \longrightarrow o$ steadily. Considering the present series as a series of the same type we have

$$a_n = \frac{2\,(2n+1)\,!}{n.\ n\,!\ (n+2)\,!}\left(\frac{c}{2r}\right)^n$$

$$=\frac{2}{n(n+2)}\cdot\frac{1.\ 3.\ 5\ldots.(2n+1).\ 2^n.\ n\,!}{n\,!\ (n+1)\,!}\left(\frac{c}{2r}\right)^n$$

$$=\frac{2^2}{n(n+2)}\cdot\frac{1.\ 3.\ 5\ldots.\ (2n+1)}{2.\ 4.\ 6\ldots.\ (2n+2)}\cdot\left(\frac{2c}{r}\right)^n$$

and this would be a decreasing monotonous sequence tending towards the limit zero if we take

$$r \geq 2c$$

$$\geq \mathrm{SS}',$$

where S′ is the second focus. We can also show by applying the usual ratio-test that the series is absolutely convergent under the same conditions. This shows that in addition to the outside region the series is also convergent inside the area lying between the previous limiting circle (drawn for the purpose of integration) and a concentric circle whose radius is SS′. Consequently the present form of the

potential function is valid at all points inside these two circles (and outside the elliptic area as we are dealing only with the external potential).

It would seem that we are incapable of accepting the potential function in the present form of the infinite series inside the circle of radius SS′. But in fact the region in which this trigonometrical series fails is much more limited. If we take S′ as our origin and proceed to find the potential by the present method we get the same series which in a similar manner may be shown to be applicable everywhere outside a circle of radius SS′. In general these two circles bounding the regions of convergence overlap outside the elliptic area and it is only inside the two small areas common to the two circles and symmetrical about the minor axis that the present trigonometrical series fails. Excepting this common portion the present form of V would hold good everywhere only we should take care to choose the origin properly—measuring r from S or S′ according as the point lies inside the circle of centre S′ or S.

It is curious to note that the convergence of the series depends on the eccentricity of the ellipse. The two limiting circles would have their common portion entirely within the elliptic area if

$$SS' \leqq SB, \text{ where } B \text{ is an extremity of the minor axis ;}$$

$$i.e., \quad 2ae \leqq a$$

$$i.e., \quad e \leqq \tfrac{1}{2}.$$

This shows that when the eccentricity of the ellipse is not greater than $\frac{1}{2}$ the function V gives the potential everywhere outside the elliptic area, with judicious choice of origin. This includes the important case when the ellipticity is small and the ellipse is obtained from a circle by a slight deformation,

For the area within the two circular strips in which the trigonometrical series fails it is not possible to get by the present method a simple value for the potential function V. Starting from the beginning, we have to divide the elliptic area into two areas by a circle passing through the point where the potential is sought such that every part of the one area is nearer to the origin than the point while every part of the second area is further from it. We can use two logarithmic expansions in the two areas and find out the potential of the two areas separately. The method of procedure is the same as

in § 9. We get the potential both in direct and inverse powers of r. But as the expression is not a simple one we do not propose to give it here.

7.

A similar investigation is possible in the case of the variable density. When the heterogeneity is of the nature we have assumed in § 4 we can show that at least outside the same two strips of areas between the two limiting circles the series V in § 4 is convergent. It should be noticed that a transfer of origin to the other focus in the case of heterogeneity would entail a change in the law of density. But if we take the density to be a rational, algebraic, integral function of the co-ordinates of a point a transfer of origin would involve a change of density of such a nature that the new distribution would still be represented by terms of the form $\rho^p \cos q\phi$ and $\rho^p \sin q\phi$. So these two cases are sufficient for our purpose.

As before applying Dirichlet's test to V in § 4 we get the condition of convergence by making the co-efficient of $\cos n\theta$ steadily tend to zero. This leads to such a condition as the following

$$\underset{n\to\infty}{Lt}\; n.\frac{(2n+p+q+1)\,!}{(n+p+2)\,!\,(n+q)\,!}\frac{e^{n+q}}{2^{n+q}}.$$

$$F\left(n+1+\frac{p+q}{2}, n+\tfrac{3}{2}+\frac{p+q}{2}, n+q+1, e^2\right)\left(\frac{l}{r}\right)^n\longrightarrow o.$$

If $p \leqq q$ every term of F is less than the corresponding term in the expansion of $\dfrac{1}{(1-e^2)^{n+\frac{3}{2}+\frac{p+q}{2}}}$, that

$$F < \frac{1}{(1-e^2)^{n+\frac{3}{2}+\frac{p+q}{2}}}.$$

Let $p>q$; the hypergeometric series is of the form

$$F\,(a, a+\tfrac{1}{2}, \gamma\,;\, e^2)=1+\frac{a(a+\frac{1}{2})}{1.\gamma}e^2+\frac{a(a+1)(a+\frac{1}{2})(a+\frac{3}{2})}{1.\,2.\,\gamma(\gamma+1)}e^4+\ldots$$

Since $a>\gamma$ (q being positive)

$$\frac{a}{\gamma}>\frac{a+1}{\gamma+1}>\frac{a+2}{\gamma+2}>\text{etc}\,;$$

so $$F < 1+\frac{a+\frac{1}{2}}{1}\cdot\left(\frac{a}{\gamma}e^2\right)+\frac{(a+\frac{1}{2})(a+\frac{3}{2})}{2\,!}\left(\frac{a}{\gamma}e^2\right)^2+\dots\dots\dots\dots$$

$$< \frac{1}{\left(1-\frac{a}{\gamma}e^2\right)^{a+\frac{1}{2}}}$$

when the series is convergent.

Here $\mathrm{Lt}\ \frac{a}{\gamma}=1$; we can show as in § 6 that V should converge at least (whatever p and q may be) if

$$\underset{n\to\infty}{\mathrm{Lt}}\quad C\times\left[\frac{2el}{r(1-e^2)}\right]^n \to 0$$

where C is ultimately of the order $\frac{1}{n^2}$; if

$$r > \frac{2le}{1-e^2}$$

i.e., $$> 2e.$$

Hence, at least outside the same restricted region as in § 6, V represents the potential function for the whole external space.

8.

In § 3 let us suppose that e is equal to zero. An ellipse of zero eccentricity is a circle and the semi-latus rectum is the radius. Making this substitution we have the logarithmic potential of a circular area

$V=\pi a^2 \log r=$(area of the circle)$\times$(log of the distance from the centre) and the potential of an infinite circular cylinder is twice this quantity neglecting an infinite constant. Similarly from § 4 when the density varies as the inverse focal distance we have

V=(circumference of the circle)$\times$(log of the distance from the centre) and the potentials of heterogeneous circular cylinders can in the same way be deduced from the other formulae in § 4. Of course all these results admit of easy verification by direct integration.

We shall deduce another simple result from the series for the potential function in § 3. Let us calculate the attraction of the elliptic cylinder at a point on the major—axis produced of the section. On the major axis

$$\frac{\partial V}{\partial \theta}=0$$

and the attraction is $2\left(\frac{\partial V}{\partial r}\right)_{\theta=0}$.

Differentiating the series of §3 we have

$$\frac{(1-e^2)^{\frac{3}{2}}}{\pi l^2}\left(\frac{\partial V}{\partial r}\right)_{\theta=0}=\frac{1}{r}+\sum_{n=1}^{\infty}(-1)^n\frac{(2n+1)\,!}{n\,!\,(n+2)\,!}\,\frac{1}{2^{n-1}}\,\frac{c^n}{r^{n+1}}$$

$$2\left(\frac{\partial V}{\partial r}\right)_{\theta=0}=4\pi ab\left[\frac{1}{r}+\sum_{n=1}^{\infty}(-1)^n\frac{2(2n+1)\,!}{n\,!\,(n+2)\,!}\left(\frac{c}{2}\right)^n\frac{1}{r^{n+1}}\right]$$

$$=4\pi\frac{ab}{c^2}r\left[\frac{1}{2\,!}\frac{c^2}{r^2}-\frac{1.3}{3\,!}\frac{c^3}{r^3}+\frac{1.3.5}{4\,!}\frac{c^4}{r^4}-\dots\quad\dots\right]$$

$$=4\pi\frac{ab}{c^2}r\left[\left(1+\frac{c}{r}\right)-\sqrt{1+2\frac{c}{r}}\right]$$

$$=4\pi\frac{ab}{c^2}\left[(c+r)-\sqrt{(c+r)^2-c^2}\right]$$

$$=4\pi\frac{ab}{c^2}\left[\xi-\sqrt{\xi^2-c^2}\right]$$

where ξ is the distance of the point from the centre of the ellipse. This is the total attraction of an infinite elliptic cylinder at an external point on the major axis of its section, a very well-known result.

It is also interesting to note that when the cylinder is heterogeneous, the density at any point of the section varying as the inverse focal distance, the attraction at any point on the major axis is similarly expressible in a very simple form, Using the corresponding formula of § 4 we have as before $\frac{\partial V}{\partial \theta}=0$ on the major axis ;

and

$$\frac{(1-e^2)^{\frac{1}{2}}}{\pi l}\left(\frac{\partial V}{\partial r}\right)_{\theta=0}=\frac{2}{r}+\sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{n!\,(n+1)!}\cdot\frac{1}{2^{n-1}}\quad\frac{\;}{r^{n+1}}$$

$$=\frac{2}{r}+\frac{2}{c}\sum_{n=1}^{\infty}(-1)^n\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n+2)}\left(\frac{2c}{r}\right)^{n+1}$$

$$=\frac{2}{r}+\frac{2}{c}\left[\left(1+\frac{2c}{r}\right)^{\frac{1}{2}}-\left(1+\frac{c}{r}\right)\right]$$

Hence the total attraction $=2\left(\frac{\partial V}{\partial r}\right)_{\theta=0}$

$$=4\pi\frac{b}{c}\left[\left(1+\frac{2c}{r}\right)^{\frac{1}{2}}-1\right]$$

$$=4\pi\frac{b}{c}\left[\sqrt{\frac{\xi+c}{\xi-c}}-1\right]$$

where ξ is the distance of the point from the centre of the ellipse.

9.

We have so far considered the case of the complete elliptic area. The method of analysis followed here is, however, applicable to the case of an area bounded by two elliptic arcs. As any two arbitrary arcs would make the results cumbrous we choose here for illustration a very simple case when the result appears in a rather symmetrical form. Let us suppose that the two elliptic arcs have the same focus and their major axes lie along the same line. Let S be the common focus and let the two arcs whose equations are

$$\rho=\frac{l}{1+e\cos\phi}$$

and
$$\rho=\frac{l'}{1+e'\cos\phi'}$$

intersect at C and D and let CS make an angle β with the line from which ϕ is measured. P is any point (ρ,ϕ) inside the area and A another point (r,θ) outside at a sufficient distance from the focus. The area

is divided into two elliptic sectors by the radii vectores SC and SD and the potential of the whole area is the sum of the potentials V_1 and V_2 due to the two sectors. We shall suppose the area to be of unit density. Then

$$V_1 = \int_{-\beta}^{\beta} \int_0^{\frac{l}{1+e\cos\phi}} \log AP \; \rho d\rho d\phi$$

$$= \int_{-\beta}^{\beta} \int_0^{\frac{l}{1+e\cos\phi}} \left[\log r - \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\rho}{r}\right)^n \cos n(\phi-\theta) \right] \rho d\rho d\phi$$

r being supposed to be greater than SC and the point A lying outside the limiting circle described with centre S and radius SC. Hence

$$V_1 = \frac{l^2}{2}\int_{-\beta}^{\beta} \frac{d\phi}{(1+e\cos\phi)^2} \log r - \sum_{n=1}^{\infty} \frac{l^{n+2}}{n(n+2)r^n} \int_{-\beta}^{\beta} \frac{\cos n(\phi-\theta)\, d\phi}{(1+e\cos\phi)^{n+2}}$$

or, $$\frac{V_1}{l^2} = I_0(\beta,e)\log r - 2\sum_{n=1}^{\infty} \frac{I_n(\beta,e)}{n(n+2)} \left(\frac{l}{r}\right)^n \cos n\,\theta,$$

where $$I_n(\beta, e) = \int_0^{\beta} \frac{\cos n\phi \, d\phi}{(1+e\cos\phi)^{n+2}}$$

and $$I_0(\beta, e) = \int_0^{\beta} \frac{d\phi}{(1+e\cos\phi)^2}$$

$$= \frac{2}{(1-e^2)^{\frac{3}{2}}} \tan^{-1}\left[\tan\frac{\beta}{2}\sqrt{\frac{1-e}{1+e}}\right] - \frac{e}{1-e^2} \cdot \frac{\sin\beta}{1+e\cos\beta}.$$

We shall put $I_n(\beta, e)$ in the form of a series. Putting

$$(1+e\cos\phi)^{-\overline{n+2}} = \sum_{m=0}^{\infty} A_m^{n+2}\cos m\phi$$

where $$A_m^{n+2} = (-1)^m \frac{(n+m+1)!}{(n+1)!\,m!} \frac{e^m}{2^{m-1}} F\left(\frac{n+m+2}{2}, \frac{n+m+3}{2}, m+1; e^2\right),$$

$(e<1)$ § 2.

$$\text{Since} \quad \int_0^\beta \cos m\phi \cos n\phi \, d\phi = \tfrac{1}{2}\left[\frac{\sin (m+n)\beta}{m+n} + \frac{\sin (m-n)\beta}{m-n}\right]$$

$$\equiv \tfrac{1}{2} S_{m,n}$$

$$I_n(\beta, e) = \int_0^\beta \frac{\cos m\phi \, d\phi}{(1+e\cos\phi)^{n+2}}$$

$$= \tfrac{1}{2} \sum_{m=0}^{\infty} A_m^{n+2} S_{m,n};$$

we have

$$\frac{V_1}{l^2} = I_0(\beta, e) \log r - \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{A_m^{n+2} S_{m,n}}{n(n+2)} \left(\frac{l}{r}\right)^n \cos n\theta.$$

Similarly

$$\frac{V_2}{l'^2} = I'_0 (\pi-\beta, e') \log r - \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^{m+1} \frac{A_m'^{n+2} S_{m,n}}{n(n+2)} \left(\frac{l'}{r}\right)^n \cos n\theta.$$

The potential of the complete area

$$V = V_1 + V_2.$$

It should be observed that the quantities l, l', β, e, e' are not all independent: in fact β is determined by the equation

$$\frac{l}{1+e\cos\beta} = \frac{l'}{1-e'\cos\beta}.$$

When the point A lies within the limiting circle an analysis on the same lines is possible if we divide the elliptic area into two parts (by a circle with S as centre and SA as radius) in which two separate logarithmic expansions would apply. In the most general case of two arbitrary elliptic arcs, the area may be looked upon as the sum of two elliptic segments each of which is the difference of an elliptic sector and a triangle. In the preceding analysis we have virtually given the potential of an elliptic sector and the potential of a triangle is known. But as the result in all these cases are not simple or symmetrical it is unnecessary to deal with them here.

10.

In this connection we may also study the potential of the complete cylinder when the density is an exponential function of the vectorial angle ϕ. As will be shown below this may be considered as a generalisation of the preceding cases. The method of analysis followed would be exactly similar.

Suppose $\sigma = e^{k\phi}$.

Then

$$V=\int_{-\pi}^{\pi}\int_{0}^{\frac{l}{1+\epsilon\cos\phi}} e^{k\phi}\left[\log r-\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\rho}{r}\right)^{n}\cos n(\phi-\theta)\right]\rho d\rho d\phi$$

$$=\frac{l^2}{2}\int_{-\pi}^{\pi}\frac{e^{k\phi}\,d\phi}{(1+\epsilon\cos\phi)^2}\log r-\sum_{n=1}^{\infty}\frac{l^{n+2}}{n(n+2)r^n}\int_{-\pi}^{\pi}\frac{e^{k\phi}\cos n(\phi-\theta)d\phi}{(1+\epsilon\cos\phi)^{n+2}}.$$

Now

$$\int_{-\pi}^{\pi}\frac{e^{k\phi}\cos n(\phi-\theta)}{(1+\epsilon\cos\phi)^{n+2}}\,d\phi=\tfrac{1}{2}\int\sum A_m^{n+2}\,e^{k\phi}\Big[\cos\,(\overline{n+m}\,\phi-n\theta)$$

$$+\cos\,(\overline{n-m}\,\phi-n\theta)\Big]\,d\phi$$

and

$$\int_{-\pi}^{\pi} e^{k\phi}\cos\,(\overline{n+m}\,\phi-n\theta)\,d\phi$$

$$=(-1)^{n+m}\,2.\,\frac{k\cos n\theta-(n+m)\sin n\theta}{k^2+(n+m)^2}\sinh k\pi,$$

and

$$\int_{-\pi}^{\pi} e^{k\phi} \cos\,(\overline{n-m}\,\phi-n\theta)\,d\phi$$

$$=(-1)^{n-m}\ 2.\ \frac{k\cos n\theta-(n-m)\sin n\theta}{k^2+(n-m)^2}\ \sinh k\pi$$

Hence

$$\frac{V}{l^2\sinh k\pi}=\sum_{m=0}^{\infty}(-1)^m\frac{2k\ A_m^2}{k^2+m^2}\log r-\sum_{n=1}^{\infty}\ \sum_{m=0}^{\infty}(-1)^{n+m}$$

$$\left[\frac{k\cos n\theta-(n+m)\sin n\theta}{k^2+(n+m)^2}+\frac{k\cos n\theta-(n-m)\sin n\theta}{k^2+(n-m)^2}\right]\times\frac{A_m^{n+2}}{n(n+2)}\left(\frac{l}{r}\right)^n.$$

If we put $k=o$ the cylinder becomes homogeneous and the present series in this limiting case degenerates into the series of § 3. Moreover the case of the density $\rho\mu e^{k\phi}$ can be easily worked out in a similar manner and putting $ik(i=\sqrt{-1})$ for k we can deduce the formulae of § 4. Thus this form appears to embody in itself all the preceding different cases.

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Notes on Inversion.

BY

TARAKNATH BHATTACHARYYA.

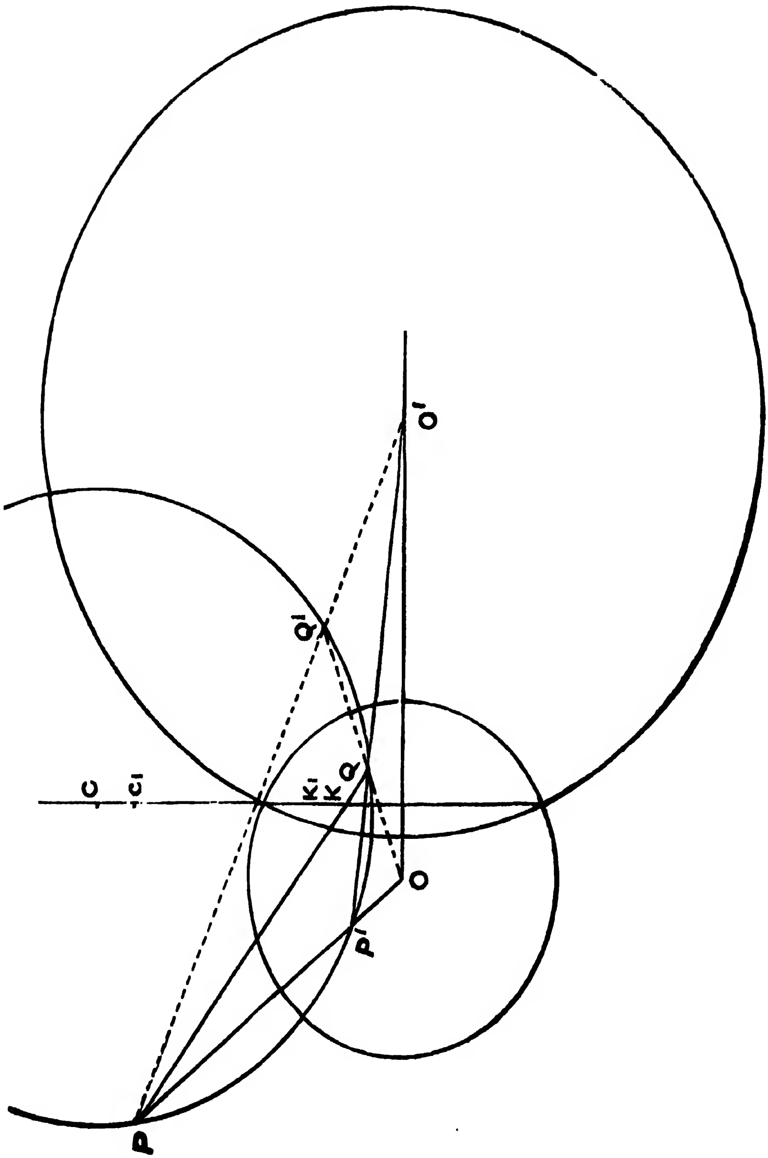
I

1. *Introductory remarks.*—The inverse of a circle with respect to any origin O is a circle; but if the circle of inversion intersects the circle orthogonally, it becomes its own inverse, the points being redistributed. Thus to find the inverse of any point P, we describe a circle through P cutting the circle of inversion orthogonally, and find the point where OP intersects this circle. Thus, "All circles that pass through a fixed point and cut a given circle orthogonally must pass through a second fixed point."

If a quadrangle be inscribed in a conic, its diagonal triangle is self-polar with respect to the conic. If however the diagonal triangle be given, an infinite number of such quadrangles can be drawn. Hence "If an infinity of triangles be inscribed in a conic so that their sides may pass through three fixed points, these fixed points will form a triangle self-polar with respect to the conic." When the conic is a circle, its centre is at the orthocentre of the triangle formed by the fixed points.

2. *Theorem.*—The inverse of a point P with respect to two given orthogonal circles in succession is a fixed point Q *which is independent of the order in which the circles are taken.*

Thus, let (O) and (O′) be two circles cutting one another orthogonally and P any point in the plane. Describe a circle (C) through P cutting the given circles orthogonally. Join OP cutting (C) in P′, and let O′P′ cut it again in Q, which is thus the inverse of P with respect to the given circles. Now PP′Q is a triangle inscribed in a circle and two of its sides pass through O and O′. Therefore, by § 1, PQ will pass through a third fixed point K, the orthocentre of the triangle COO′. Thus Q must be necessarily determinate. Similar remarks apply for the triangle PQQ′. Hence the inverse of a point with respect to two orthogonal circles in succession, taken twice over, is the point itself.


O'
Q'
C
C'
K'
K
Q
O
P'
P

3. We may go a step further. We can invert with respect to *four* mutually orthogonal circles and the result will be *the same*. But if four circles cut one another orthogonally, the centre of any one of them is (the radical centre and hence) the orthocentre of the triangle formed by the centres of the other three circles. One of these circles, therefore, must be imaginary. Thus a glance at the adjoining figure is sufficient to reveal the theorem : "The inverse of a point P with respect to the four mutually orthogonal circles O, O′, C, K in succession is the original point itself : " The theorem, as proved in this way, is seen to be true only when the point P lies on one of the four circles. To prove it in all generality we need only recall that Q is determinate so long as C and K are fixed and that we can replace any pair of orthogonal circles, such as O and O′, by another orthogonal pair belonging to the same coaxal system, without affecting the final point Q.

II

1. A conic may be regarded as the envelope of a variable tangent. Thus the inverse of a conic with respect to the origin O (which may be easily proved to be a nodal bicircular quartic, the common points of the variable circles being nodes on the curve) is the envelope of a variable circle passing through O. The locus of the centres of these circles is clearly a conic which is called the Focal Conic of the quartic.

2. *Theorem.*—If a system of co-axal circles intersect in two real points O and O′, and if through one of these O, two straight lines are drawn to cut the system at P, P′, P″, ..., Q, Q′, Q″,......, the envelope of the straight lines PQ, P′Q′, P″Q″, ... will be a parabola of which the other point O′ will be the focus.

For, since O′ is a point on every circle, the pedal line L of O′ with respect to the triangles OPQ, OP′Q′, ... will evidently be the same. Therefore, PQ, P′Q′, P″Q″, ... will all envelope the first negative pedal of L with respect to O′, which is a parabola having O′ for the focus and L the tangent at the vertex. Clearly, the parabola also touches each of the given straight lines.

3. Now invert the whole figure with respect to O′. Then we have the theorem : " If two circles cut in two real points O and O′,

and if through one of these O an infinity of straight lines POQ, P′OQ′, P″OQ″, ... be drawn, then the circumcircles of the triangles O′PQ, O′P′Q′, O′P″Q″,. ...... envelop a cardioide touching the given circles.

Since when a straight line inverts into a circle, the image of the origin in that straight line inverts into the centre of that circle, we see that the centres of the circumscribing circles lie on another circle passing through the origin O′. This is, therefore, the Focal Circle of the cardioide. If the given circles are orthogonal, the focal circle will also pass through O.

Hence *the directrix of the parabola inverts into the focal circle of the cardioide.*

The above theorem may be stated in various ways; thus,—

"If the three angles of a triangle are given while the vertex is fixed and the base passes through a fixed point, the circumscribing circle of the triangle envelops a cardioide, and its centre describes a circle passing through the vertex and through the fixed point if the vertical angle be a right angle.

4. Next let us invert the theorem of § 2 with respect to O. Let us, moreover, suppose the given lines through O to be at right angles. We thus find the following theorem. "If a system of straight lines be drawn through a point O′ to cut a given pair of perpendicular lines through O, at the points PQ, P′Q′,......, then the envelope of the circumcircles OPQ, OP′Q′,..... will be a bicircular quartic having a cusp at O."*

Further, it is seen that since O was on the directrix of the parabola, the focal conic of the quartic will be a rectangular hyperbola whose asymptotes are parallel to the given lines. Hence also, the elementary theorem: "The locus of the middle points of the segments intercepted between two given perpendicular straight lines of any number of straight lines drawn through a given point is a rectangular hyperbola passing through that point, of which the asymptotes are parallel to the given straight lines

* The quartic will touch the given lines at their points of intersection with straight lines through O′ parallel to them.

III

1. The correspondence between successive inversions and rotations is established in the classical memoirs of (among others) Klein, Cayley and Poincare. We shall here only make a few remarks on the Cross Ratio Group of Projective Geometry,—

$$z, \ \frac{1}{z}, \ 1-z, \ \frac{1}{1-z}, \ \frac{z-1}{z}, \ \frac{z}{z-1}.$$

2. It is well-known that the non-homogeneous substitutions of the Dihedral Group are—

$$z' = e^{\frac{2ik\pi}{n}} z, \qquad z' = \frac{e^{\frac{2ik\pi}{n}}}{z}$$

which are derivable from Cayley's formula [see, *e.g.*, Forsyth Theory of Functions, ch. xxii, Art. 300].

The cross ratio group in question is nothing other than the Dihedral group for $n=3$.

3. *Lemma.*—The direction of the axis being the same, if the origin be transferred to a point O′ (f, g, h) and if all the points of the points of the z—plane are centrally projected from (0, 0, 1) to the new plane of z', to find the relation between the corresponding values of z, z'.

It can be seen without difficulty that the relation in question is

$$z' = (1-h)z - (f+ig).$$

This transformation will be real if $g=0$, when we have

$$z' = (1-h)z - f \qquad \ldots \quad (a)$$

4. Now take the case of the Dihedron in which $n=3$.

The polygon here will therefore be an equilateral triangle. Take its plane to be the plane of $\eta=0$. Let the summits be called A, B, C,—where C, the point (0, 0, 1) is also the vertex of projection.

We are now to choose the new origin O′. Since the cross ratio group permutes the numbers 0, 1, ∞ among themselves, it is clear that the transformation must be made in which the three summits correspond to the numbers 0, 1, ∞ on the axis of real numbers.

Hence, for instance, as the point A is to project to zero, the point O′ must be in CA. Thus the origin must lie in the plane of $\eta=0$. And since the point B is to become by projection the point 1, the length O′ B′ (B′ being the point where CB cuts the new x—axis) must equal unity. Hence without difficulty, the coordinates of O′ are seen to be $\left(-\frac{1}{2},\ 0,\ 1-\sqrt{\frac{3}{2}}\right)$. Here since $f=0$, the transformation is necessarily real and equation (a) of Lemma gives

$$z'=\frac{\sqrt{3}.\ z+1}{2},\ \text{or}\ z=\frac{2z'-1}{\sqrt{3}}.$$

The rotations belonging to the dihedral group are:—

(i) the rotation about a line through O, the centre of the sphere, perpendicular to the plane of the paper, (a) through 120°, (b) through 240°, (c) through 360°,—this last giving the identical substitution ;

(ii) the rotation through 180° about each of the secondary axes through A, B and C.

Using Cayley's formula and then making the transformation here indicated, the corresponding substitutions are seen to be in the following order.

$$z'=\frac{z-1}{z},\quad z'=\frac{1}{1-z}\ \text{and}\ z'=z;$$

$$z'=\frac{z}{z-1},\quad z'=\frac{1}{z},\ \text{and}\ z'=1-z.$$

Thus we have obtained all the substitutions of this group, and the six anharmonic ratios of four points in a straight line thus have a correspondence with the rotations of a sphere, or with their equivalents, the successive inversions in circles. Interpretations of the harmonic and equianharmonic groups are now easy and at the same time interesting.

On the use of Ritz's method for finding the vibration-frequencies of heterogeneous strings and membranes.

BY

N. K. MAJUMDAR.

CONTENTS:

Introduction.

1. The object of the paper is to show how reliable results about the vibration-frequencies of *heterogeneous* strings and membranes can be obtained by the use of a method, the germs of which are found in Lord Rayleigh's writings, and which was first clearly expounded by Ritz.*

2. It is believed that no previous writer has applied this methop to determine the vibration-frequencies of *heterogeneous* strings and membranes, although the method has found applications to numerous other problems by many investigators, including Ritz himself, who considered the vibration of plates,† Prof. A. E. H. Love, who studied the theory of tides,‡ Prof. Kalahne and Dr. Reinstein.

* *Crelle's Journal*, Vol. CXXXV.

† *Annalen der Physik*, Vol. XXVIII, 1909.

‡ *Fifth International Congress of Mathematicians*, Vol. II, 1912.

Ritz's method.

3. If it is required to render the integral

$$J \equiv \int_{x_0}^{x_1} f_1(x, y, y', y'', \dots y^{(\sigma)})dx$$

an extremum under the isoperimetric condition

$$I \equiv \int_{x_0}^{x_1} f_2(x, y, y', y'', \dots y^{(\sigma)})dx = \text{constant}.$$

the problem is equivalent to that of rendering $J+\lambda I$ an extremum without any isoperimetric condition.

It is well known that y must satisfy, as a necessary condition, some differential equation $D=0$, although every solution of $D=0$ may not render $J+\lambda I$ an extremum.

Conversely, if a solution of $D=0$ is required, which satisfies certain boundary conditions, and if we can obtain the corresponding isoperimetrical problem of the calculus of variations, the required solution of the differential equation may be taken any 'y' which renders $J+\lambda I$ an extremum.

Ritz's method consists in obtaining successive approximations to the value of y by the following process:—

Substitute for y in the integral $J+\lambda I \equiv J'$ say, $Y_n \equiv y_0 + a_1 y_1 + \dots + a_n y_n$, where $y_0, y_1, y_2, \dots y_n$ are known functions, $a_1, a_2, \dots a_n$ are constants to be determined from the condition of rendering $J+\lambda I$ an extremum, and Y_n satisfies the proscribed boundary conditions.

By this substition J' becomes a known function $J_n(a_1, a_2, \dots a_n)$ of the a's, independent of x. The a's are determined so that J_n may be an extremum, *i.e.*, from the n equations

$$\frac{\partial J_n}{\partial a_r} = 0, \ (r=1, 2, \dots n) \qquad \dots \qquad \dots \qquad (A)$$

In the boundary value problems of mathematical physics, we have to deal mostly with linear differential equations. J_n is thus a function of the second degree in the a's, and the equations (A) are therefore linear in the a's. There exists thus one and only one solution of the system (A), and we get the following successive approximations to the value of Y—

$$Y_1 = y_0 + a_1 y_1,$$

$$Y_2 = y_0 + a_1 y_1 + a_2 y_2,$$

etc., etc.

Heterogeneous Strings.

4. If $\rho =$ density, the equation of motion is:

$$\rho \frac{d^2 w}{dt^2} = \frac{d^2 w}{dx^2}$$

on putting $w = \cos kt \cdot y$, this reduces to the ordinary differential equation

$$\frac{d^2 y}{dx^2} + k^2 \rho y = 0;$$

the boundary conditions being, say, $y(\pm 1) = 0$.

Any y will satisfy this differential equation, if it renders the integral

$$J' \equiv \int_{-1}^{+1} y'^2 \, dx$$

an extremum, and at the same time

$$I \equiv \int_{-1}^{+1} \rho y^2 \, dx = 1;$$

i.e., if it renders

$$J \equiv J' - k^2 I \equiv \int_{-1}^{+1} (y'^2 - k^2 p y^2)\, dx$$

an extremum without any isoperimetric condition.

4. CASE I: $\rho = 1 + qx^2$. *First* Approximation.

Put for y in J, $y_1 \equiv ((1-x^2)\ (a_0 + a_1 x^2))$, (which satisfies the necessary boundary conditions).

$$\text{Then} \quad J_1 = \int_{-1}^{+1} [y'^2 - k^2\ (1+qx^2) y^2]\, dx$$

$$= \int_{-1}^{+1} \Big[\{ (a_1 - a_0) - 2a_1 x^2 \}^2\ (4x^2)$$

$$- k^2\ (1+qx^2)\ (1-x^2)^2\ (a_0 + a_1 x^2)^2 \Big] dx$$

The system of equations, $\dfrac{\partial J_1}{\partial a_0} = 0$, $\dfrac{\partial J_1}{\partial a_1} = 0$, on the elimination of a_0 and a_1, leads to the following equation for the determination of k^2—

$$k^4 \left[1 + \frac{6}{11} q + \frac{1}{33} q^2\right] - k^2 \left[28 + \frac{48}{11} q\right] + 63 = 0,$$

which agrees with Ritz's equation,

$$k^4 - 28k^2 + 63 = 0,$$

for the case $q = 0$.

If q is considered to be very small, neglecting q^2 in the solution of the above equation, the first approximation to the fundamental-tone is given by the least root:

$$2k_1^2 = 4{\cdot}93\ (1 - q \times {\cdot}134).$$

According to Lord Rayleigh's formula*

$$2k_1^2=\frac{\pi^2}{2}(1-q\theta),$$

where

$$\theta=\int_0^2 (x-1)^2 \sin^2 \frac{\pi x}{2}\, dx = \cdot 131.$$

5. CASE I : $\rho=1+qx^2$. *Second* approximation.

In the definite integral, put

$$Y\equiv(1-x^2)\,(a_0+a_1 x^2+a_2 x^4),$$

when we get

$$J_2\equiv\int_{-1}^{+1}[\{a_0+a_1(2x^2-1)+a_2(3x^4-2x^2)\}(4x^2)$$

$$-k^2(1+qx^2)\,(1-x^2)^2\,(a_0+a_1 x^2+a_2 x^4)^2]dx.$$

The elimination of a_0, a_1, a_2 from the system

$$\frac{\partial J_2}{\partial a_r}=0,\ (r=0,1,2)$$

leads to the following equation for the determination of $\lambda=2k^2$, neglecting q^2 and higher powers, *viz.*,

$$38610-\lambda\,(8910+1422q)+\lambda^2\,(225+114q)-\lambda^3\,(1+q)=0,$$

whence for the fundamental note we have

$$2k_1^2=\lambda_1=4\cdot 9348\,(1-\cdot 1306q)\text{ approx.}$$

7. CASE II : $\rho=1+q\cos 2x$. First approximation.

Substituting in J

$$Y\equiv(1-x^2)\,(a_0+a_1 x^2),$$

* Lord Rayleigh, *Theory of Sound*, Vol. I, p. 118.

we get

$$J_1 \equiv \int_{-1}^{+1} [\{a_0+a_1(2x^2-1)\}^2 (4x^2)$$

$$-k^2(1+q\cos 2x)(1-x^2)^2(a_0+a_1x^2)^2]dx.$$

If $q\cos 2=8p$, the elimination of a_0 and a_1 from $\frac{\partial J_1}{\partial a_0}=0$ and $\frac{\partial J_1}{\partial a_1}=0$

leads to the following equation for $2k^2$:

$$k^4\left[16-210p\left(3549+\frac{3247}{2}t\right)\right]$$

$$-k^2\left[448-315p\left(5997+\frac{5471}{2}t\right)\right]+1008=0$$

neglecting p^2 and higher powers, as q and hence p is supposed to be very small; also $t=\tan 2=-2\frac{1}{6}$ nearly. This equation again tallies with Ritz's equation

$$k^4-28k^2+63=0$$

for the case $q=0$.

The fundamental note, as a first approximation, is given by the least root

$$2k_1^2=4\cdot 93\,(1-\cdot 656q).$$

According to Lord Rayleigh,

$$2k_1^2=\frac{\pi^2}{2}(1-q\theta),$$

where

$$\theta=\frac{\pi^2\sin 2}{2(\pi^2 4)}=\cdot 75 \text{ nearly}.$$

Heterogeneous square membrane.

8. The equation of motion is

$$\rho\frac{d^2w}{dt^2}=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2};$$

putting

$$w = V \cdot \cos 2kt,$$

we get

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + 4k^2 \rho V = 0,$$

where V satisfies some prescribed boundary conditions, say,

$$(V)_{x=\pm 1} = 0, \ (V)_{y=\pm 1} = 0.$$

Here we must render

$$J' \equiv \int_{-1}^{+1} \int_{-1}^{+1} \left\{ \left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 \right\} dx dy$$

an extremum, under the isoperimetric condition

$$I \equiv \int_{-1}^{+1} \int_{-1}^{+1} \rho V^2 dx dy = \text{constant}.$$

which is equivalent to rendering

$$J \equiv J' - k^2 I \equiv \int_{-1}^{+1} \int_{-1}^{+1} \left[\left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 - 4k^2 \rho V^2 \right] dx dy$$

an extremum without any isoperimetric condition.

By Ritz's method $V(x, y)$ is to be found as successive approximations to

$$V_{m,n} \equiv (1-x^2)(1-y^2)[a_{0,0} + (a_{1,0} x^2 + a_{0,1} y^2) + (a_{2,0} x^4 + a_{1,1} x^2 y^2 + a_{0,2} y^4) + \ldots].$$

9. Case I $\rho = 1$. *Homogeneous* square membrane. First approximation.

As a first approximation, we substitute for V in J

$$V_{1,1} \doteq (1-x^2)(1-y^2)(a + bx^2 + cy^2).$$

The elimination of a, b, c from

$$\frac{\partial J_{1,1}}{\partial a} = 0, \quad \frac{\partial J_{1,1}}{\partial b} = 0, \quad \frac{\partial J_{1,1}}{\partial c} = 0,$$

leads to the following equation

$$(k^2-7)\ (32k^4-264k^2+277)=0,$$

the roots of which are

$$k_1{}^2=1\cdot 234.$$

$$k_2{}^2=7.$$

$$k_3{}^2=7\cdot 01, \text{ nearly}.$$

10. CASE II : $\rho=1+qx^2y^2$ (*Heterogeneous*). First approximation.

Substituting V_{111} in J as before, and eliminating a, b, c from the equations

$$\frac{\partial J_{111}}{\partial a}=\frac{\partial J_{111}}{\partial b}=\frac{\partial J_{111}}{\partial c}=0,$$

we get the following equation for k^2 ;

$$\left[7-k^2\left(1+\frac{q}{3\times 7\times 11}\right)\right]+\left[k^4\left\{32-\frac{1058\times 16}{11\times(21)^2}q-\frac{16\times 680}{(21)^5\times 11}q^2\right\}\right.$$

$$\left.-k^2\left\{264-\frac{4\times 3332}{(21)^2\times 11}q\right\}+277\right]=0.$$

Considering q to be small, and neglecting q^2 and higher powers, the roots are

$$k_1{}^2=1\cdot 234(1-q\times\cdot 008),$$

$$k_2{}^2=7(1-q\times\cdot 004),$$

$$k_3{}^2=7\cdot 01(1+q\times\cdot 11).$$

Before I conclude I must express my deep sense of gratitude and indebtedness to Dr. Ganesh Prasad who kindly suggested to me the problem and helped me with other directions.

On the steady motion of a viscous fluid due to the rotation of two rigid bodies about arbitrary axes

BY

BIJON DUTT.

The present paper is the first instalment of the results of my investigation on the mutual influence between any two given bodies capable of rotating about any two given axes in a viscous fluid medium. The simplest case of this problem, *viz.*, that in which the given bodies are spheres capable of rotating about the line through their centres, has been recently studied by Mr. G. B. Jeffery.*

In the present paper, I have given the complete solution for two more cases, *viz.*, the case of two spheroids rotating about a common axis and two cylinders rotating about two parallel axes. Other cases are also being studied by me and these will be given in a subsequent paper.

I wish to express my obligation to Dr. S. K. Banerji at whose suggestion I took up and under whose guidance I carried on the investigation.

Case I. *Two Spheroids rotating about a Common Axis.*

Suppose that a point P has the polar coordinates (r, θ) and (r', θ') referred to two points O and O′, θ and θ' being measured in opposite senses from the line OO′, and let OO′$=c$.

* "On the steady motion of a solid of revolution in a viscous fluid." (*Proceedings of the London Mathematical Society*, February, 1915).

Then the following transformation theorems are well-known:—

$$\text{(I).}\quad \frac{P'^{m}_{n}}{r'^{n+1}} = \frac{r^{m}}{(n-m)!\,c^{n+m+1}}\left[\frac{(n+m)!}{2m!}P^{m}_{m} + \frac{(n+m+1)!}{(2m+1)!}\frac{r}{c}P^{m}_{m+1} + \dots + \frac{(n+m+s)!}{(2m+s)!}\left(\frac{r}{c}\right)^{s}P^{m}_{m+s} + \dots\right]$$

(if $r < c$),

$$\text{(II).}\quad \frac{P^{m}_{n}}{r^{n+1}} = \frac{r'^{m}}{(n-m)!\,c^{n+m+1}}\left[\frac{(n+m)!}{2m!}P'^{m}_{m} + \frac{(n+m+1)!}{(2m+1)!}\frac{r'}{c}P'^{m}_{m+1} + \frac{(n+m+s)!}{(2m+s)!}\left(\frac{r'}{c}\right)^{s}P'^{m}_{m+s} + \dots\right]$$

(if $r' < c$),

$P^{m}_{n}(\cos\theta)$ being an associated Legendre function of degree n and order m whose origin is O and the axis is OO′ and $P'^{m}_{n}(\cos\theta')$ a similar function whose origin is O′ and the axis is O′O.

The equations of the spheroids with centres at O and O′ and with OO′ as the common axis of revolution, can be written as

$$r = a\,[1 + \epsilon\, P_2(\cos\theta)], \qquad \dots \quad (1)$$

$$r' = b\,[1 + \epsilon'\, P'_2(\cos\theta')], \qquad \dots \quad (2)$$

where ϵ and ϵ' are two small quantities whose second and higher powers are supposed to be negligible.

Let ω and ω' denote the angular velocities of the spheroids.

Mr. Jeffery has shown* that if (ρ, z, ϕ) are a set of cylindrical coordinates and if we have a solution of Laplace's equation of the form

$$f(\rho, z)\sin\phi, \qquad \dots \quad (3)$$

then the solution of a slow, steady and symmetrical motion about the axis of z is given by

$$v = f(\rho, z), \qquad \dots \qquad \dots \quad (4)$$

where v is the velocity at any point of the fluid.

* *Proceedings of the London Mathematical Society*, February, 1915.

Now it is well known that

$$r^{-n-1}\, P^1{}_n\,(\cos\theta)\sin\phi \quad \ldots \quad \ldots \quad (5)$$

is a solution of Laplace's Equation.

We accordingly assume as the solution of our present problem

$$v=\sum_{n=1}^{\infty}\left[\frac{A_n}{r^{n+1}}\,P^1{}_n(\cos\theta)+\frac{B_n}{r'^{n+1}}\,P'{}_n{}^1(\cos\theta')\right], \quad \ldots \quad (6)$$

A_n, B_n being arbitrary constants to be determined.

This expression for v apparently vanishes at infinity. It remains to determine the constants so that the following two conditions may also be satisfied

$$v=\omega r\sin\theta, \qquad \text{when } r=a[1+\epsilon P_2\,(\cos\theta)]. \quad \ldots \quad (7)$$

$$v=\omega' r'\sin\theta', \qquad \text{when } r'=b[1+\epsilon' P'_2\,(\cos\theta')]. \quad \ldots \quad (8)$$

Putting $m=1$ in "Theorem I," we get

$$\frac{P'{}_n{}^1}{r'^{n+1}}=\frac{r}{(n-1)!\,c^{n+2}}\left[\frac{(n+1)!}{2!}\,P_1{}^1+\frac{(n+2)!}{3!}\,\frac{r}{c}\,P_2{}^1+\ldots\right.$$

$$\left.+\frac{(n+s+1)!}{(s+2)!}\left(\frac{r}{c}\right)^s P_{s+1}{}^1+\ldots\right]. \quad \ldots \quad (9)$$

Therefore

$$v=\sum_{n=1}^{\infty}\frac{A_n}{r^{n+1}}\,P_n{}^1\,(\cos\theta)+\sum_{k=1}^{\infty}\frac{R_{k+1}}{k+1}\,\frac{r^k}{c^{k+1}}\,P_k{}^1\,(\cos\theta), \quad \ldots \quad (10)$$

where

$$R_{k+1}=(k+1)\,\frac{B_1}{c}+\frac{(k+1)(k+2)}{1!}\,\frac{B_2}{c^2}+\frac{(k+1)(k+2)(k+3)}{2!}\,\frac{B_3}{c^3}$$

$$+\ldots \quad \ldots \quad (11)$$

We note the following well known properties of Legendre's co-efficients :—

$$P_n^1(\mu)=(1-\mu^2)^{\frac{1}{2}}\frac{dP_n(\mu)}{d\mu}, \quad \ldots \quad \ldots \quad \ldots (12)$$

$$\frac{dP_n(\mu)}{d\mu}=(2n-1)\ P_{n-1}(\mu)+(2n-5)\ P_{n-3}(\mu)$$

$$+(2n-9)\ P_{n-5}(\mu)+\ldots, \quad \ldots (13)$$

the last term being $P_0(\mu)$ or $3P_1(\mu)$ according as n is odd or even

$$(\mu^2-1)\frac{dP_n(\mu)}{d\mu}=n\mu P_n(\mu)-nP_{n-1}(\mu). \quad \ldots (14)$$

$$nP_n(\mu)-(2n-1)\ \mu P_{n-1}(\mu)+(n-1)\ P_{n-2}(\mu)=0 \quad \ldots (15)$$

Hence

$$P_2(\mu)P_n^1(\mu)=\frac{3\mu^2-1}{2}(1-\mu^2)^{\frac{1}{2}}\frac{dP_n(\mu)}{d\mu} \quad \text{[by (12)}$$

$$=\left\{\frac{3}{2}(\mu^2-1)+1\right\}(1-\mu^2)^{\frac{1}{2}}\frac{dP_n(\mu)}{d\mu}$$

$$=(1-\mu^2)^{\frac{1}{2}}\left[\frac{3}{2}\{n\mu P_n(\mu)-nP_{n-1}(\mu)\}\right.$$

$$\left.+\{(2n-1)P_{n-1}(\mu)+(2n-5)P_{n-3}(\mu)+\ldots\}\right]$$

[by (13), (14).

Therefore using (15), we have

$$P_2(\mu)P_n^1(\mu)=(1-\mu^2)^{\frac{1}{2}}\left[\frac{3}{2}\frac{n(n+1)}{2n+1}\{P_{n+1}(\mu)-P_{n-1}(\mu)\}\right.$$

$$\left.+\{(2n-1)P_{n-1}(\mu)+(2n-5)P_{n-3}(\mu)+\ldots\}\right]\ldots (16)$$

Now

$$r=a[1+\epsilon P_2(\cos\theta)].$$

Therefore neglecting the second and higher powers of ϵ,

$$r^k = a^k[1+k\epsilon P_2(\cos\theta)], \qquad \ldots \qquad \ldots (17)$$

$$\frac{1}{r^{n+1}} = \frac{1}{a^{n+1}}[1-(n+1)\epsilon P_2(\cos\theta)]. \qquad \ldots (18)$$

Therefore on writing μ for $\cos\theta$,

$$r^k P_k{}^1(\cos\theta) = a^k[1+k\epsilon P_2(\mu)]P_k{}^1(\mu);$$

$$r^k P_k{}^1(\cos\theta) = a^k\Big[\{(2n-1)P_{k-1}(\mu)+(2n-5)P_{k-3}(\mu)+\ldots$$

$$+\epsilon k\left\{\frac{3}{2}\frac{k(k+1)}{2k+1}\{P_{k+1}(\mu)-P_{k-1}(\mu)\}+(2k-1)P_{k-1}(\mu)\right.$$

$$\left.+(2k-5)P_{k-3}(\mu)+\ldots\right\}\Big] \quad \ldots (19)$$

and

$$\frac{P_n{}^1(\cos\theta)}{r^{n+1}} = \frac{1}{a^{n+1}}\{1-(n+1)\epsilon P_2(\mu)\}P_n{}^1(\mu)$$

$$\frac{P_n{}^1(\cos\theta)}{r^{n+1}} = \frac{1}{a^{n+1}}(1-\mu^2)^{\frac{1}{2}}\Big[\{(2n-1)P_{n-1}(\mu)+(2n-5)P_{n-3}(\mu)$$

$$+\ldots\}-\epsilon(n+1)\left\{\frac{3}{2}\frac{n(n+1)}{2n+1}\{P_{n+1}(\mu)-P_{n-1}(\mu)\}\right.$$

$$\left.+(2n-1)P_{n-1}(\mu)+(2n-5)P_{n-3}(\mu)+\ldots\right\}\Big]. \quad \ldots (20)$$

Also

$$\omega r \sin\theta = wa[1+\epsilon P_2(\cos\theta)](1-\cos^2\theta)^{\frac{1}{2}}$$

or,

$$\omega r \sin\theta = wa(1-\mu^2)^{\frac{1}{2}}[1+\epsilon P_2(\mu)] \qquad \ldots \qquad \ldots (21)$$

Therefore the condition (7) is equivalent to

$$\omega a^2[1+\epsilon P_2(\mu)] = \sum_{n=1}^{\infty} \frac{A_n}{a^n}\Big[\{(2n-1)\,P_{n-1}(\mu)+(2n-5)P_{n-3}(\mu)+\dots\}$$

$$-\epsilon\,(n+1)\left[\frac{3}{2}\,\frac{n(n+1)}{2n+1}\{P_{n+1}(\mu)-P_{n-1}(\mu)\}+(2n-1)\,P_{n-1}(\mu)+\dots\}\right]$$

$$+\sum_{k=1}^{\infty}\frac{R_{k+1}}{k+1}\left(\frac{a}{c}\right)^{k+1}\Big[\{(2k-1)P_{k-1}(\mu)+(2k-5)P_{k-3}(\mu)+\ \dots\}$$

$$+\epsilon k\left\{\frac{3}{2}\,\frac{k(k+1)}{2k+1}\{P_{k+1}(\mu)-P_{k-1}(\mu)\}+(2k-1)P_{k-1}(\mu)+\dots\right\}\Big] \quad (22)$$

Equating the co-efficients of $P_0(\mu)$ from both sides of this identity, we get

$$\omega a^2=\left(\frac{A_1}{a}+\frac{A_3}{a^3}+\frac{A_5}{a^5}+\dots\right)-\epsilon\left(\frac{4A_3}{a^3}+\frac{6A_5}{a^5}+\dots\right)$$

$$+\left(\frac{a^2}{c^2}\,\frac{R_2}{2}+\frac{a^4}{c^4}\,\frac{R_4}{4}+\dots\right)+\epsilon\left(\frac{3a^4}{c^4}\,\frac{R_4}{4}+\frac{5a^6}{c^6}\,\frac{R_6}{6}+\dots\right) \quad (23)$$

Equating the co-efficients of $P_2(\mu)$ similarly, we get

$$\omega a^2\epsilon=5\left(\frac{A_3}{a^3}+\frac{A_5}{a^5}+\dots\right)-5\epsilon\left(\frac{4A_3}{a^3}+\frac{6A_5}{a^5}+\dots\right)$$

$$+5\left(\frac{a^4}{c^4}\,\frac{R_4}{4}+\frac{a^6}{c^6}\,\frac{R_6}{6}+\dots\right)+5\epsilon\left(\frac{3a^4}{c^4}\,\frac{R_4}{4}+\frac{5a^6}{c^6}\,\frac{R_6}{6}+\dots\right)$$

$$-\epsilon\,\frac{2A_1}{a}+\epsilon\,\frac{72}{7}\,\frac{A_3}{a^3}+\epsilon\frac{a^2}{c^2}\,\frac{R_2}{2}-\epsilon\,\frac{27}{14}\,\frac{a^4}{c^4}\,R_4 \quad \dots \quad \dots \quad (24)$$

Hence by subtraction,

$$\omega a^2\,(5-\epsilon)=\frac{A_1}{a}\,(5+2\epsilon)-\epsilon\,\frac{72}{7}\,\frac{A_3}{a^4}+\frac{a^2}{2c^2}\,(5-\epsilon)\,R_2+\epsilon\,\frac{27}{14}\,\frac{a^4}{c^4}\,\frac{R_4}{4} \quad (25)$$

Neglecting terms of orders higher than that of $\frac{a^2}{c^2}$ this becomes

$$\omega a^2 (5-\epsilon)=\frac{A_1}{a}(5+2\epsilon) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

$$\therefore \quad A_1=\omega a^3\left(\frac{5-\epsilon}{5+2\epsilon}\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

Consequently by symmetry,

$$B_1=\omega' b\left(\frac{5-\epsilon'}{5+2\epsilon'}\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

Thus to this order of approximation

$$v=\omega a^3\left(\frac{5-\epsilon}{5+2\epsilon}\right)\frac{P_1{}^1(\cos\theta)}{r^2}+\omega' b^3\left(\frac{5-\epsilon'}{5+2\epsilon'}\right)\frac{P'_1{}^1(\cos\theta')}{r'^2} \quad \ldots \quad (29)$$

For higher approximations we proceed as follows:—

Equating the co-efficients of $P_n(\mu)$ from both sides of the identity (22), we get for $n>0$ and $\neq 2$

$$0=(2n+1)\left(\frac{A_{n+1}}{a^{n+1}}+\frac{A_{n+3}}{a^{n+3}}+\ldots\right)-\epsilon(2n+1)\left(\frac{n+2}{a^{n+1}}A_{n+1}\right.$$

$$\left.+\frac{n+4}{a^{n+3}}A_{n+3}+\ldots\right)$$

$$+(2n+1)\left\{\left(\frac{a}{c}\right)^{n+2}\frac{R_{n+2}}{n+2}+\left(\frac{a}{c}\right)^{n+4}\frac{R_{n+4}}{n+4}+\ldots\right\}$$

$$+\epsilon(2n+1)\left\{\frac{n+1}{n+2}\left(\frac{a}{c}\right)^{n+2}R_{n+2}+\frac{n+3}{n+4}\left(\frac{a}{c}\right)^{n+4}R_{n+4}+\ldots\right.$$

$$-\epsilon\frac{3n^2(n-1)}{2(2n-1)}\frac{A_{n-1}}{a^{n-1}}+\epsilon\frac{3(n+1)(n+2)^2}{2(2n+3)}\frac{A_{n+1}}{a^{n+1}}$$

$$+\epsilon\frac{3(n-1)^2}{2(2n-1)}\left(\frac{a}{c}\right)^{n}R_n-\epsilon\frac{3(n+1)^2}{2(2n+3)}\left(\frac{a}{c}\right)^{n+2}R_{n+2} \quad \ldots \quad (30)$$

Similarly equating the co-efficients of $P_{n+2}(\mu)$, we get

$$0=(2n+5)\left(\frac{A_{n+3}}{a^{n+3}}+\frac{A_{n+5}}{a^{n+5}}+\ldots\right)-\epsilon(2n+5)\left(\frac{n+4}{a^{n+3}}A_{n+3}+\frac{n+6}{a^{n+5}}A_{n+5}+\ldots\right)$$

$$+(2n+5)\left\{\left(\frac{a}{c}\right)^{n+4}\frac{R_{n+4}}{n+4}+\left(\frac{a}{c}\right)^{n+6}\frac{R_{n+6}}{n+6}+\ldots\right\}$$

$$+\epsilon(2n+5)\left\{\frac{n+3}{n+4}\left(\frac{a}{c}\right)^{n+4}R_{n+4}+\frac{n+5}{n+6}\left(\frac{a}{c}\right)^{n+6}R_{n+6}+\ldots\right\}$$

$$-\epsilon\frac{3(n+2)^2(n+1)}{2(2n+3)}\frac{A_{n+1}}{a^{n+1}}+\epsilon\frac{3(n+3)(n+4)^2}{2(2n+7)}\frac{A_{n+3}}{a^{n+3}}$$

$$+\epsilon\frac{3(n+1)^2}{2(2n+3)}\left(\frac{a}{c}\right)^{n+2}R_{n+2}-\epsilon\frac{3(n+3)^2}{2(2n+7)}\left(\frac{a}{c}\right)^{n+4}R_{n+4} \qquad (31)$$

Multiplying (30) by $(2n+5)$ and (31) by $(2n+1)$ and subtracting we get

$$0=\frac{A_{n+1}}{a^{n+1}}\left[(2n+1)(2n+5)-(n^2+3n-1)(n+2)\epsilon\right]$$

$$-\epsilon\frac{A_{n-1}}{a^{n-1}}\frac{3n^2(n-1)(2n+5)}{2(2n-1)}-\epsilon\frac{A_{n+3}}{a^{n+3}}\frac{3(n+3)(n+4)^2(2n+1)}{2(2n+7)}$$

$$+\frac{R_{n+2}}{n+2}\left(\frac{a}{c}\right)^{n+2}\left[(2n+1)(2n+5)+\epsilon(n^2+3n-1)(n+1)\right]$$

$$+\epsilon R_n\left(\frac{a}{c}\right)^n\frac{3(n-1)^2(2n+5)}{2(2n-1)}+\epsilon\left(\frac{a}{c}\right)^{n+4}R_{n+4}\frac{3(n+3)^2(2n+1)}{2(2n+7)} \qquad (32)$$

Neglecting the terms containing A_{n+3} and R_{n+4} we have

$$0=\frac{A_{n+1}}{a^{n+1}}\left[(2n+1)(2n+5)-(n^2+3n-1)(n+2)\epsilon\right]$$

$$-\epsilon\frac{A_{n-1}}{a^{n-1}}\frac{3n^2(n-1)(2n+5)}{2(2n-1)}$$

$$+\frac{R_{n+2}}{n+2}\left(\frac{a}{c}\right)^{n+2}\left[(2n+1)(2n+5)+(n^2+3n-1)(n+1)\epsilon\right]$$

$$+\epsilon\left(\frac{a}{c}\right)^n R_n\frac{3(n-1)^2(2n+5)}{2(2n-1)} \qquad \ldots \qquad (33)$$

Putting $n-2$ for n in this equation, we get

$$0=\frac{A_{n-1}}{a^{n-1}}\left[(2n-3)(2n+1)-n(n^2-n-3)\epsilon\right]$$

$$-\epsilon\frac{A_{n-3}}{a^{n-3}}\frac{3(n-2)^2(n-3)(2n+1)}{2n-5}$$

$$+\frac{a^n}{c^n}\frac{R_n}{n}\left[(2n-3)(2n+1)+(n+1)(n^2-n-3)\epsilon\right]$$

$$+\frac{a^{n-2}}{c^{n-2}}\epsilon R_{n-2}\frac{3(n-3)^2(2n+1)}{2(2n-5)} \quad \ldots \quad (34)$$

Substituting the value of A_{n-1} from (34) in (33) and neglecting ϵ^2, we have

$$0=\frac{A_{n+1}}{a^{n+1}}\left[(2n+1)(2n+5)-(n^2+3n-1)(n+2)\epsilon\right]$$

$$+\left(\frac{a}{c}\right)^{n+2}\frac{R_{n+2}}{n+2}\left[(2n+1)(2n+5)+(n^2+3n-1)(n+1)\epsilon\right]$$

$$+\left(\frac{a}{c}\right)^n R_n\,\epsilon\,\frac{3}{2}(n-1)(2n+5) \quad \ldots \quad \ldots \quad (35)$$

By symmetry

$$0=\frac{B_{n+1}}{b^{n+1}}\left[(2n+1)(2n+5)-(n^2+3n-1)(n+2)\epsilon'\right]$$

$$+\left(\frac{b}{c}\right)^{n+2}\frac{S_{n+2}}{n+2}\left[(2n+1)(2n+5)+(n^2+3n-1)(n+1)\epsilon'\right]$$

$$+\left(\frac{b}{c}\right)^n S_n\,\epsilon'\,\frac{3}{2}(n-1)(2n+5), \quad \ldots \quad \ldots \quad (36)$$

where

$$S_{k+1}=(k+1)\frac{A_1}{c}+\frac{(k+1)(k+2)}{1!}\frac{A_2}{c^2}$$

$$\frac{(k+1)(k+2)(k+3)}{2!}\frac{A_3}{c_3}+\ldots \quad \ldots \quad (37)$$

Therefore, for $n > 0$ and $\neq 2$,

$$A_{n+1} = -\frac{a^{2n+3}}{c^{n+2}} \frac{R_{n+2}}{n+2} \left[1 + \frac{(n^2+3n-1)(2n+3)\epsilon}{(2n+1)(2n+5)-(n^2+3n-1)(n+2)\epsilon}\right]$$

$$-\frac{a^{2n+1}}{c^n} R_n \epsilon \frac{3(n-1)(2n+5)}{2\{(2n+1)(2n+5)-(n^2+3n-1)(n+2)\epsilon\}} \qquad (38)$$

$$B_{n+1} = -\frac{b^{2n+3}}{c^{n+2}} \frac{S_{n+2}}{n+2} \left[1 + \frac{(n^2+3n-1)(2n+3)\epsilon'}{(2n+1)(2n+5)-(n^2+3n-1)(n+2)\epsilon'}\right]$$

$$-\frac{b^{2n+1}}{c^n} S_n \epsilon' \frac{3(n-1)(2n+5)}{2\{(2n+1)(2n+5)-(n^2+3n-5)(n+2)\epsilon'\}} \qquad (39)$$

To determine A_3 and B_3, we proceed as follows :—

On equating the co-efficients of $P_4(\mu)$ from both sides of (22), we have

$$0 = 9\left[\frac{A_5}{a^5} + \frac{A_7}{a^7} + \ldots\right] - 9\epsilon\left[6\frac{A_5}{a^5} + 8\frac{A_7}{a^7} + \ldots\right]$$

$$+9\left[\frac{a^6}{c^6}\frac{R_6}{6} + \frac{a^8}{c^8}\frac{R_8}{8} + \ldots\right] + 9\epsilon\left[\frac{5}{6}\frac{a^6}{c^6}R_6 + \frac{7}{8}\frac{a^8}{c^8}R_8 + \ldots\right]$$

$$-\epsilon\frac{72}{7}\frac{A_3}{a^3} + \epsilon\frac{273}{11}\frac{A_5}{a^5} + \epsilon\frac{27}{14}\frac{a^4}{c^4}R_4 - \epsilon\frac{75}{22}\frac{a^6}{c^6}R_6 \quad \ldots \qquad (40)$$

From (40) and (24), we get

$$9\omega a^2\epsilon = 9\frac{A_3}{a^3}(5-8\epsilon) - \epsilon\frac{18A_1}{a} + \frac{a^4}{c^4}R_4\frac{9}{4}(5-3\epsilon) + \epsilon\frac{9}{2}\frac{a^2}{c^2}R_2$$

$$+\epsilon\frac{325}{22}\frac{a^6}{c^6}R_6 - \epsilon\frac{1350}{11}\frac{A^5}{a^5} \quad \ldots \qquad (41)$$

Neglecting the terms containing R_4, R_6 and A_5, we have

$$9\omega a^2\epsilon = 9\frac{A_3}{a^3}(5-8\epsilon) - \epsilon\frac{18A_1}{a} + \frac{9}{2}\epsilon\frac{a^2}{c^2}R_2 \quad \ldots \quad \ldots \qquad (42)$$

Now

$$A_1=\omega a^3\left(\frac{5-\epsilon}{5+2\epsilon}\right),\quad B_2=2\,\frac{\omega' b^3}{c}\left(\frac{5-\epsilon'}{5+2\epsilon'}\right)$$

Therefore

$$A_3=3\omega a^5\left(\frac{\epsilon}{5-8\epsilon}\right)-\frac{\omega' a^3 b^3}{c^3}\left(\frac{5-\epsilon'}{5+2\epsilon'}\right)\frac{\epsilon}{5-8\epsilon} \qquad \ldots \quad (43)$$

Similary

$$B_3=3\omega' b^5\left(\frac{\epsilon'}{5-8\epsilon'}\right)-\frac{\omega a^3 b^5}{c^3}\left(\frac{5-\epsilon}{5+2\epsilon}\right)\left(\frac{\epsilon'}{5-8\epsilon'}\right) \qquad \ldots \quad (44)$$

On calculating A_2 and B_2 by means of formula (38) and (39) we find

$$A_2=-\omega'\,\frac{a^5 b^3}{c^4}\left(\frac{5-\epsilon'}{5+2\epsilon'}\right)\left\{1+\frac{5\epsilon}{7-3\epsilon}\right\}$$

$$B_2=-\omega\,\frac{a^3 b^5}{c^4}\left(\frac{5-\epsilon}{5+2\epsilon}\right)\left\{1+\frac{5\epsilon'}{7-3\epsilon'}\right\}$$

Case II. *Two Circular Cylinders rotating about Parallel Axes.*

We will suppose the fluid to be incompressible and the external forces to be derivable from a potential V.

Then writing

$$\chi=-V-\frac{p}{\rho} \qquad \ldots \quad (1)$$

the equations of motion in two dimensions are

$$\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}&=\frac{\partial \chi}{\partial x}+\nu\nabla^2 u,\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}&=\frac{\partial \chi}{\partial y}+\nu\nabla^2 v,\end{aligned} \qquad \ldots \quad (2)$$

where ν stands for the kinematic coefficient of viscosity. Putting $\frac{\partial}{\partial t}=0$ for steady motion and neglecting the squares and products of velocity for slow motion, these equations reduce to

$$\left.\begin{aligned}\nabla^2 u&=-\frac{1}{\nu}\frac{\partial\chi}{\partial x}\\ \nabla^2 v&=-\frac{1}{\nu}\frac{\partial\chi}{\partial y}\end{aligned}\right\}\quad\ldots\quad(3)$$

On eliminating χ we get

$$\nabla^2\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)=0\quad(4)$$

The equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\quad\ldots\quad(5)$$

Consequently there exists a function ψ of x and y, such that

$$u=-\frac{\partial\psi}{\partial y},\ v=\frac{\partial\psi}{\partial x}\quad\ldots\quad(6)$$

Hence (4) becomes

$$\nabla^4\psi=0.\quad\ldots\quad(7)$$

We take co-ordinates defined by

$$x+iy=c\tan\frac{1}{2}(\xi+i\eta).\quad(8)$$

Whence

$$x=\frac{c\sin\xi}{\cos\xi+\cosh\eta},\ y=\frac{c\sinh\eta}{\cos\xi+\cosh\eta},\quad\ldots\quad(9)$$

and

$$x^2+y^2-2cy\coth\eta+c^2=0\quad\ldots\quad(10)$$

Equation (10) represents two families of circles whose centres lie on the axis of y at distances $\pm c \coth \eta$ from the origin, and whose radii are equal to $c \operatorname{cosech} \eta$. These circles do not cut the axis of x.

We can choose the axes of reference and the constant c, so that any two given non-intersecting circles are members of these families.

We assume the rotating cylinders to be given by

$$\eta = \alpha, \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

$$\eta = -\beta. \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

We have

$$\nabla^2 = \frac{1}{AB}\left[\frac{\partial}{\partial \xi}\left(\frac{B}{A}\frac{\partial}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{A}{B}\frac{\partial}{\partial \eta}\right)\right], \quad \ldots \quad (13)$$

where

$$ds^2 = A^2 d\xi^2 + B^2 d\eta^2. \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

From (9), we find

$$ds^2 = \frac{c^2}{(\cos \xi + \cosh \eta)^2}\,(d\xi^2 + d\eta^2), \quad \ldots \quad (15)$$

so that

$$A = B = \frac{c}{\cos \xi + \cosh \eta} \quad \ldots \quad \ldots \quad (16)$$

Hence (13) reduces to

$$\nabla^2 = \frac{1}{c^2}\,(\cos \xi + \cosh \eta)^2 \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) \quad \ldots \quad (17)$$

Writing

$$\nabla_1^2 \text{ for } \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2},$$

the equation (7) takes the form

$$\nabla_1^2[(\cos\xi+\cosh\eta)^2\nabla_1^2\psi]=0 \quad \ldots \quad \ldots \quad (18)$$

To solve this equation we shall write

$$\psi=\phi\theta \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

and assume

$$\nabla_1^2\phi=0. \quad \ldots \quad (20)$$

Then (19) becomes

$$\nabla_1^2[(\cos\xi+\cosh\eta)^2\nabla_1^2(\phi\theta)]=0 \quad \ldots \quad (21)$$

But

$$\nabla_1^2(\phi\theta)=\theta\nabla_1^2\phi+2\left(\frac{\partial\phi}{\partial\xi}\frac{\partial\theta}{\partial\xi}+\frac{\partial\phi}{\partial\eta}\frac{\partial\theta}{\partial\eta}\right)+\phi\nabla_1^2\theta \quad \ldots \quad (22)$$

Remembering that $\nabla_1^2\phi=0$, (21) then becomes

$$\nabla_1^2\left[(\cos\xi+\cosh\eta)^2\left\{2\left(\frac{\partial\phi}{\partial\xi}\frac{\partial\theta}{\partial\xi}+\frac{\partial\phi}{\partial\eta}\frac{\partial\theta}{\partial\eta}\right)\right.\right.$$

$$\left.\left.+\phi\nabla_1^2\theta\right\}\right] \ldots \quad (23)$$

First, we assume

$$\theta=\frac{\cos\xi}{\cos\xi+\cosh\eta},$$

so that

$$\frac{\partial\theta}{\partial\xi}=\frac{-\sin\xi\cosh\eta}{(\cos\xi+\cosh\eta)^2},\quad \frac{\partial\theta}{\partial\eta}=\frac{-\cos\xi\sinh\eta}{(\cos\xi+\cosh\eta)^2},$$

$$\nabla_1^2\theta=-\frac{2}{(\cos\xi+\cosh\eta)^2}$$

and (23) becomes

$$\nabla_1^2\left[-2\sin\xi\cosh\eta\frac{\partial\phi}{\partial\xi}-2\cos\xi\sinh\eta\frac{\partial\phi}{\partial\eta}-2\phi\right]=0. \quad (24)$$

But

$$\nabla_1^2\left(\sin\xi\cosh\eta\frac{\partial\phi}{\partial\xi}\right)=\frac{\partial\phi}{\partial\eta}\nabla_1^2(\sin\xi\cosh\eta)$$

$$+2\frac{\partial}{\partial\xi}(\sin\xi\cosh\eta)\frac{\partial^2\phi}{\partial\xi^2}+2\frac{\partial}{\partial\eta}(\sin\xi\cosh\eta)\frac{\partial^2\phi}{\partial\xi\partial\eta}$$

$$+\sin\xi\cosh\eta\,\nabla_1^2\frac{\partial\phi}{\partial\xi}$$

$$=2\cos\xi\cosh\eta\frac{\partial^2\phi}{\partial\xi^2}+2\sin\xi\sinh\eta\frac{\partial^2\phi}{\partial\xi\partial\eta},$$

$$\nabla_1^2\left(\cos\xi\sinh\eta\frac{\partial\phi}{\partial\eta}\right)=+2\cos\xi\cosh\eta\frac{\partial^2\phi}{\partial\eta^2}$$

$$-2\sin\xi\sinh\eta\frac{\partial^2\phi}{\partial\xi\partial\eta}.$$

Therefore remembering that $\nabla_1^2\phi=0$, the left hand side of (24) becomes

$$-4\cos\xi\cosh\eta\left(\frac{\partial^2\phi}{\partial\eta^2}+\frac{\partial^2\phi}{\partial\eta^2}\right)$$

which is zero since $\nabla_1^2\phi=0$.

Thus (23) is satisfied by this assumption for θ. Accordingly

$$\frac{\cos\xi}{\cos\xi+\cosh\eta}\phi$$

is a solution of (18).

We can similarly show that

$$\frac{\sin\xi}{\cos\xi+\cosh\eta}\phi,\ \frac{\cosh\eta}{\cos\xi+\cosh\eta}\phi \text{ and } \frac{\sinh\eta}{\cos\xi+\cosh\eta}\phi$$

are all solutions of (18).

Now

$$\begin{matrix}\cos \\ \sin\end{matrix} n\xi \begin{matrix}\cosh \\ \sinh\end{matrix} n\eta$$

are well known solutions of $\nabla_1{}^2\phi=0$.

Hence

$$\frac{\begin{matrix}\cos \\ \sin\end{matrix} (n+1)\xi \begin{matrix}\cosh \\ \sinh\end{matrix} n\eta}{\cos\xi+\cosh\eta}$$

and

$$\frac{\begin{matrix}\cos \\ \sin\end{matrix} n\xi \begin{matrix}\cosh \\ \sinh\end{matrix} (n+1)\eta}{\cos\xi+\cosh\eta}$$

are solutions of (18).

We accordingly make the following assumption for ψ.

$$\psi=\frac{1}{\cos\xi+\cosh\eta}\sum_{n=0}^{\infty}\sin\left(n+\tfrac{1}{2}\right)\xi\left[A_n\frac{\sinh\left(n-\frac{1}{2}\right)(\eta-a)}{\sinh\left(n-\frac{1}{2}\right)(a+\beta)}\right.$$

$$+B_n\frac{\sinh\left(n+\frac{3}{2}\right)(\eta-a)}{\sinh\left(n+\frac{3}{2}\right)(a+\beta)}+C_n\frac{\cosh\left(n-\frac{1}{2}\right)(\eta-a)}{\cosh\left(n-\frac{1}{2}\right)(a+\beta)}$$

$$\left.+D_n\frac{\cosh\left(n+\frac{3}{2}\right)(\eta-a)}{\cosh\left(n+\frac{3}{2}\right)(a+\beta)}\right], \quad \ldots \quad (25)$$

where A_n, B_n, C_n, D_n are arbitrary constants which are to be determined so that the boundary conditions may be satisfied.

The velocity component tangential to the curve $\xi=$constant is

$$\frac{\partial\psi}{A\partial\xi} \text{ or } \frac{\cos\xi+\cosh\eta}{c}\frac{\partial\psi}{\partial\xi},$$

and that tangential to the curve $\eta=$constant is

$$\frac{\partial\psi}{B\partial\eta} \text{ or } \frac{\cos\xi+\cosh\eta}{c}\frac{\partial\psi}{\partial\eta}.$$

Therefore on the hypothesis of no slipping, the conditions to be satisfied on the surfaces of the cylinders are

$$(i) \quad \frac{\partial\psi}{\partial\xi}=0, \text{ when } \eta=\alpha,$$

$$(ii) \quad \frac{\partial\psi}{\partial\xi}=0, \text{ when } \eta=\beta,$$

$$(iii) \quad \frac{\cos\xi+\cosh\eta}{c}\cdot\frac{\partial\psi}{\partial\eta}=\omega a, \text{ when } \eta=\alpha,$$

$$(iv) \quad \frac{\cos\xi+\cosh\eta}{c}\cdot\frac{\partial\psi}{\partial\eta}=\omega' b, \text{ when } \eta=-\beta,$$

where ω, ω' are the angular velocities and a, b the radii of the cylinders.

The coefficient of $\sin(n+\frac{1}{2})\xi$ in (25) reduces to

$$\frac{C_n}{\cosh(n-\frac{1}{2})(\alpha+\beta)}+\frac{D_n}{\cosh(n+\frac{3}{2})(\alpha+\beta)} \text{ and } -A_n-B_n+C_n+D_n \text{ when}$$

we put $\eta=\alpha$ and $\eta=-\beta$ respectively.

We make

$$\frac{C_n}{\cosh(n-\frac{1}{2})(\alpha+\beta)}+\frac{D_n}{\cosh(n+\frac{3}{2})(\alpha+\beta)}=0 \text{ and } -A_n-B_n+C_n+D_n=0$$

and then (i) and (ii) are directly satisfied.

Thus, we have

$$\frac{C_n}{\cosh(n-\frac{1}{2})(\alpha+\beta)}=\frac{D_n}{-\cosh(n+\frac{3}{2})(\alpha+\beta)}$$

$$=\frac{A_n+B_n}{\cosh(n-\frac{1}{2})(\alpha+\beta)-\cosh(n+\frac{3}{2})(\alpha+\beta)} \quad \cdots \quad (26)$$

Conditions (*iii*) and (*iv*) give the relations

$$\omega a=\frac{1}{c}\sum_{n=0}^{\infty}\sin\left(n+\tfrac{1}{2}\right)\xi\left[\frac{(n-\frac{1}{2})\,A_n}{\sinh(n-\frac{1}{2})(\alpha+\beta)}\right.$$

$$\left.+\frac{(n+\frac{3}{2})\,B_n}{\sinh(n+\frac{3}{2})(\alpha+\beta)}\right]\quad\ldots\quad(27)$$

$$\omega' b=\frac{1}{c}\sum_{n=0}\sin\left(n+\tfrac{1}{2}\right)\xi\left[(n-\tfrac{1}{2})\coth(n-\tfrac{1}{2})(\alpha+\beta)\,A_n\right.$$

$$+(n+\tfrac{3}{2})\coth(n+\tfrac{3}{2})(\alpha+\beta)\,B_n-(n-\tfrac{1}{2})\tanh(n-\tfrac{1}{2})(\alpha+\beta)\,C_n$$

$$\left.-(n+\tfrac{3}{2})\tanh(n+\tfrac{3}{2})(\alpha+\beta)\,D_n\right]\ldots\quad(28)$$

Having regards to (26) these can be written as

$$\omega a c=\sum_{n=0}(\lambda_n A_n+\mu_n B_n)\sin\left(n+\tfrac{1}{2}\right)\xi,\quad\ldots\quad(29)$$

$$\omega' b c=\sum_{n=0}(\lambda'_n A_n+\mu'_n B_n)\sin\left(n+\tfrac{1}{2}\right)\xi,\quad\ldots\quad(30)$$

where

$$\left.\begin{aligned}
\lambda_n&=\frac{n-\frac{1}{2}}{\sinh(n-\frac{1}{2})(\alpha+\beta)},\\
\mu_n&=\frac{(n+\frac{3}{2})}{\sinh(n+\frac{3}{2})(\alpha+\beta)},\\
\lambda'_n&=\frac{-(n-\frac{1}{2})+(n+\frac{1}{2})\cosh 2(\alpha+\beta)-\cosh(2n+1)(\alpha+\beta)}{2\sinh(\alpha+\beta)\sinh(n+\frac{1}{2})(\alpha+\beta)},\\
\mu'_n&=\frac{(n+\frac{3}{2})-(n+\frac{1}{2})\cosh 2(\alpha+\beta)-\cosh(2n+1)(\alpha+\beta)}{2\sinh(\alpha+\beta)\sinh(n+\frac{1}{2})(\alpha+\beta)}.
\end{aligned}\right\}\ldots\quad(31)$$

We can always expand a constant in a Fourier's series; we have in fact

$$k=\frac{4k}{\pi}\left(\frac{\sin\frac{\xi}{2}}{1}+\frac{\sin\frac{3\xi}{2}}{3}+\frac{\sin\frac{5\xi}{2}}{5}+\ldots\right)\quad\ldots\quad(32)$$

Therefore in order to satisfy (29) and (30), we make

$$\lambda_n A_n + \mu_n B_n = \frac{4\omega ac}{\pi} \frac{1}{2n+1} \qquad \ldots \quad (33)$$

$$\lambda'_n A_n + \mu'_n B_n = \frac{4\omega' bc}{\pi} \frac{1}{2n+1} \qquad \ldots \quad (34)$$

Therefore

$$A_n = \frac{4c}{(2n+1)\pi} \frac{\omega a \mu'_n - \omega' b \mu_n}{\lambda_n \mu'_n - \lambda'_n \mu_n} \qquad \ldots \quad (35)$$

$$B_n = \frac{4c}{(2n+1)\pi} \frac{\omega' b \lambda_n - \omega a \lambda'_n}{\lambda_n \mu'_n - \lambda_n \mu_n} \qquad \ldots \quad (36)$$

New Methods in the Geometry of a Plane Arc

Cyclic points and normals.

BY

S. Mukhopadhyaya.

Introductory.

A *simple arc* for the purposes of the present paper will be defined as follows :

(*i*) It is a continuous curve bounded by two extreme points.

(*ii*) It has a tangent at each point which turns continuosly along the arc in the same direction.

(*iii*) No straight line meets it at more than two points.

(*iv*) The circle determined by any three points of the arc is always finite and varies continuously with the determining points, in other words, the arc possesses continuous curvature.

A *simple oval* may be defined to be a closed curve of which every arc is simple.

An arc NPQ of a circle C intersecting a simple arc S at P will be said to *in*-cross S at P if it crosses from the convex to the concave side at P, and to *ab*-cross S at P if it passes from the concave to the convex side.

A circle C' is said to have *ordinary* contact with S at P if it passes through only two consecutive points of S at P. A circle having ordinary contact with S at P will be said to have *in*-contact with S at P if it falls on the concave side of S at P and to have *ab*-contact with S if it falls on the convex side of S at P.

A circle C passing through only three consecutive points at P will be said to have *cru*-contact with S at P.

If NPQ be an arc of a circle having *cru*-contact with S at P, then NPQ will be said to *in*-cross or *ab*-cross S at P according as NPQ passes from convex to concave side or from concave to convex side of S at P. If NPQ in-crosses S at P then we may say that the *portion* NP has *ab*-contact with S at P and the *portion* PQ has *in*-contact with S at P.

If a circle C passes through four consecutive points of S at P then P is called a cyclic point of S and the circle C may be said to have cyclic contact with S at P. A cyclic point will be called *in*-cyclic or *ab*-cyclic[1] according as the circle C falls on the concave or convex side of S at P.

It may be observed here that according to our New Methods a circle cannot meet a *fixed* curve at more than four consecutive points. This matter will be discussed in a subsequent paper.

We will denote an arc of S between P_1 and P_2 by S_{12} and an arc of a circle C from P_1 to P_2 by C_{12} and so on.

A circular arc C_{12} will be called *cyclic to* S between P_1 and P_2 if it meets S in two or more points between P_1 and P_2. One or two or even three of these extra points may be consecutive to P_1 or P_2 giving rise to an ordinary contact or a cru-contact or a cyclic contact of C_{12} with S at P_1 or P_2.

A circular arc C_{12} which is cyclic to S between P_1 and P_2 will be either *in*-cyclic or *ab*-cyclic or *cru*-cyclic to S between P_1 and P_2. If C_{12} produced *in*-crosses S both at P_1 and P_2 it is *in*-cyclic. If C_{12} produced *ab*-crosses S at both P_1 and P_2 it is *ab*-cyclic. If C_{12} produced *in*-crosses at one and *ab*-crosses at the other of the two points P_1 and P_2 it is *cru*-cyclic.

If C_{12} has ordinary contact or cyclic contact with S at P_1 or P_2 or at both of these points then instead of *in*-crossing or *ab*-crossing at these points we shall read *in*-contact or *ab*-contact in the above definitions.

A fundamental theorem which has been established in my first paper and of which we shall make frequent use in the present paper may now be re-stated in the following form :

If a circular arc C_{12} is in-cyclic to a simple arc S between P_1 and P_2 then there exists at least one in-cyclic point on S between P_1 and P_2. If a circular arc C_{12} is ab-cyclic to S between P_1 and P_2 then there exists at least one ab-cyclic point on S between P_1 and P_2. If a circular arc C_{12} is

[1] In my first paper (See Bulletin Calcutta Mathematical Society Vol. I. Part I), I have, it is believed for the first time, distinguished the two kinds of cylic points and called them *in*-cyclic and *ex*-cyclic. The latter kind of cyclic points I have preferred to call here *ab*-cyclic, as the prefix *ab*-seems to me more euphonic and significant.

cru-cyclic to S between P_1 and P_2 then there exists at least one in-cyclic and one ab-cyclic point on S between P_1 and P_2.

It may be observed that according to our New Methods there cannot be such a thing as a cru-cylic point on S, that is, a point at which a circle meets S at five consecutive points. It may also be mentioned in connection with the above theorem that when we say a cyclic point exists on S between P_1 and P_2 we mean to exclude P_1 and P_2.

THEOREM I.

P_1, P_2, P_3 are three points taken in order on a simple arc S and the normals at P_1, P_2, P_3 meet at a common point O, which is not the centre of curvature of S at P_2 and which is towards the concave side of S. Then there will be at least one cyclic point X on S between P_1 and P_3, provided none of the angles $P_1\,O\,P_2$ and $P_2\,O\,P_3$ exceed two right angles. The point X will be *in*-cyclic or *ab*-cyclic according as $O\,P_2$ is a *maximal* or *minimal* normal, that is, according as $O\,P_2$ is a maximum or minimum radius vector from O to S.

Case I. When each of the angles $P_1\,O\,P_2$ and $P_2\,O\,P_3$ is less than two right angles.

We may suppose without any loss of generality that $O\,P_1$ and $O\,P_3$ are the two normals from O to S, nearest to $O\,P_2$ on either side, for if X lie between the feet of two nearer normals on either side, much more will it lie between the feet of two further normals on either side.

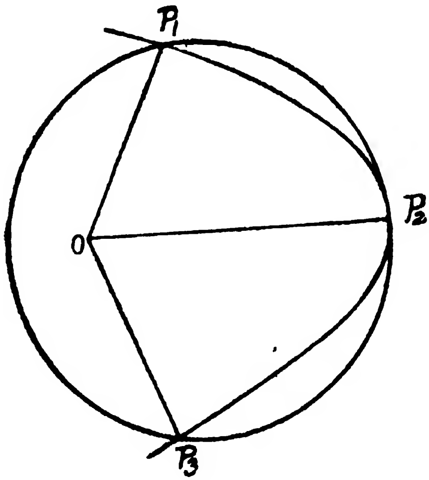
Suppose $O\,P_2$ is a maximal normal. Then $O\,P_2$ is the maximum radius vector from O to S in the whole neighbourhood $P_1\,P_2\,P_3$ and is therefore smaller than both $O\,P_1$ and $O\,P_3$. Draw a circle through P_1 to touch S at P_2. We will denote this circle by C and the arc of this circle from P_1 to P_2 by C_{12}. Then since angle $P_1\,O\,P_2$ is less than two right angles and $O\,P_1$ is less than $O\,P_2$ the arc C_{12} meets $O\,P_1$ at an obtuse angle and therefore when produced towards P_1 will *in*-cross S at P_1.

Similarly draw a circle C′ through P_3 to touch S at P_2. Denote the arc of this circle from P_3 to P_2 by C'_{32}. Then C'_{32} will meet $O\,P_3$ at an obtuse angle and therefore when produced towards P_3 will *in*-cross S at P_3.

Then either C and C′ will coincide or one will fall within the other.

If C and C′ coincide then the circular arc $P_1 P_2 P_3$ will meet S *in*-cyclically between P_1 and P_3 and therefore there must be at least one *in*-cyclic point on S between P_1 and P_3. *See fig. I.*

Fig. I.



If C and C′ do not coincide, then one will fall within the other. Suppose C falls within C′.

The circle C will have either *in*-contact or *ab*-contact or *cru*-contact with S at P_2.

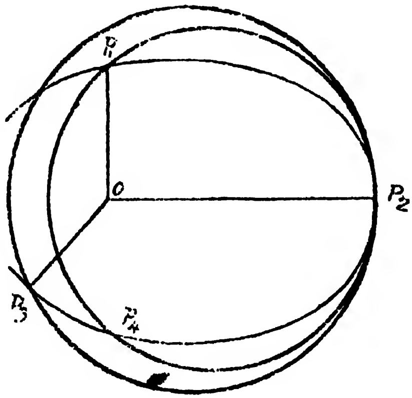
If C has *in*-contact with S at P_2 then C_{12} must cross S_{12} somewhere between P_1 and P_2, and consequently C_{12} will meet S_{12} *in*-cyclically between P_1 and P_2. Thus there is an *in*-cyclic point on S between P_1 and P_2.

If C has *ab*-contact with S at P_2 then C_{12} produced towards P_3 will pass between S_{23} and C'_{23}, *i.e.* C will enter at P_2 the space bounded by S_{23} and C'_{23}. C must therefore come out of this space at some point P_4 (*See fig. II*) on S_{23} between P_2 and P_3. Thus C meets S *in*-cyclically between P_1 and P_4. Consequently there is an *in*-cyclic point on S between P_1 and P_4.

If C has *cru*-contact with S at P_2 then C_{12} will either *in*-cross S at P_2 or *ab*-cross S at P_2. In the former case there will be an *in*-cyclic point on S between P_1 and P_2 and in the latter case an *in*-cyclic point between P_2 and P_4.

Next suppose that OP_2 is a minimal normal. In this case we can prove, by reasoning exactly similar that there is at least one *ab*-cyclic point on S between P_1 and P_3.

Fig. II.

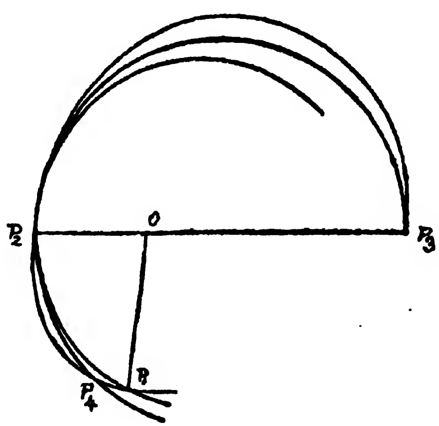


Case II. When angle $P_1 O P_2$ is less than two right angles and angle $P_2 O P_3$ is equal to two right angles.

Suppose OP_2 is a minimal normal so that OP_1 and OP_3 are each greater than OP_2.

Draw a circle C to pass through P_1 and to touch S at P_2. Then because the angle $P_1 OP_2$ is less than two right angles and OP_1 is greater than OP_2 the arc C_{12} will meet OP at an acute angle and consequently C_{12} when produced towards P_1 will *ab*-cross S at P_1. *See fig. III.*

Fig. III.



The circle C will either have *ab*-contact or *in*-contact or *cru*-contact with S at P_2. If C have *ab*-contact with S at P_2 then C will cross S between P_1 and P_2 and consequently there will be an *ab*-cyclic point on S between P_1 and P_2. If C have *cru*-contact with S at P_2 then C_{12} will either *ab*-cross S at P_2 or *in*-cross S at P_2. In the latter case C_{12} must cross S between P_1 and P_2. So that in either case there will be an *ab*-cyclic point on S between P_1 and P_2.

If C have *in*-contact with S at P_2 then C will either meet S_{23} between P_2 and P_3 at some point P_4 or fall below S_{23}. In the former case there is an *ab*-cyclic point on S between P_1 and P_4.

In the latter case draw the circle C' or rather the semi-circular arc C'_{32} to touch S at P_2 and P_3. If C'_{32} have *ab*-contact with S at P_2 and P_3 then an *ab*-cyclic point on S between P_2 and P_3 is assured. If C'_{32} have contacts *ab* and *in* or *in* and *ab* at P_2 and P_3 then C'_{32} must necessarily cross S between P_2 and P_3 and an *ab*-cyclic point on S between P_2 and P_3 is assured.

If C'_{32} have *in*-contact with S at P_2 and P_3 then C' will enter the space formed by S_{12} and C_{12} at P_2 and consequently *ab*-cross S at some point P_4 between P_1 and P_2. *See fig III* Consequently there will be an *ab*-cyclic point on S between P_3 and P_1.

Thus on the supposition that OP_2 is a minimal normal there is always an *ab*-cyclic point on S between P_1 and P_3.

If we had supposed OP_2 to be a maximal normal we could prove by similar reasoning that there is always an *in*-cyclic point on S between P_1 and P_3.

Corollary to Theorem I.

If the normals to S at P_1 and P_3 meet at P_2 then there is at least one ab-cyclic point on S between P_1 and P_3. If the normals at P_1 and P_2 meet at P_3 then there is an ab-cyclic or in-cyclic point on S between P_1 and P_3 according as P_2P_3 is a minimal or a maximal normal.

This corollary follows from theorem I if we make one of the three normals OP_1, OP_2, OP_3 vanish. It can however be proved directly quite easily.

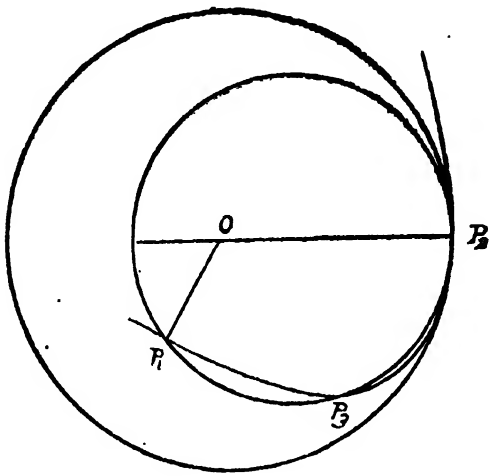
Theorem II.

If OP_1 and OP_2 be two successive normals to a simple arc S from a point O, on the concave side of S, including between them an angle not

exceeding two right angles, and if O be the centre of curvature of S at P_2, then there is at least one cyclic point on S between P_1 and P_2, which is in- or ab- according as OP_1 is less or greater than OP_2.

Suppose angle $P_1 OP_2$ is less than two right angles and OP_1 is less than OP_2. *See fig. IV.*

Fig. IV.



Draw a circle C to pass through P_1 and touch S at P_2. Then the arc C_{21} of this circle will meet OP_1 at an obtuse angle and consequently *in*-cross S at P_1.

Draw a circle C′ with centre O and radius OP_2. Then C′ is the circle of curvature of S at P_2 and touches C externally at P_2. The circular arc C_{21} will therefore have *in*-contact with S at P_2. Consequently C_{21} must cross S at some point P_3 between P_1 and P_2. Thus C_{21} is *in*-cyclic to S between P_1 and P_2 which ensures the existence of an *in*-cyclic point on S between P_1 and P_2.

If we suppose the angle P_1 OP_2 to be equal to two right angles, then C_{12} will have in contact with S at P_2 and either *in*- or *ab*-contact with S at P_1. In the former case C_{12} is in-cyclic to S between P_1 and P_2 and in the latter case C_{12} is cru-cyclic to S between P_1 and P_2. In either case the existence of an in-cyclic point on S between P_1 and P_2 is assured.

If OP_1 be greater than OP_2 the existence of an *ab*-cyclic point on S between P_1 and P_2 can be similarly established.

In this theorem we have supposed O to be the centre of curvature of S at P_2. The centre of curvature of S at P_1 will in general not be at O but it can be also at O as a special case.

Corollary to Theorem II.

If the centre of curvature of S at a point P_1 be a point P_2 which is on S then there is at least one in-cyclic point on S between P_1 and P_2.

The three following theorems follow at once from theorems I and II and their corollaries.

Theorem III.

If from a point O on the concave side of a simple arc S it is possible to draw n normals to S and if the angle between any pair of successive normals do not exceed two right angles then there are at least $n-2$ cyclic points on S between the feet of the first and the last normal.

Theorem IV.

If from a point O interior to a simple oval it is posible to draw n normals to the oval and if the angle between any pair of successive normals do not exceed two right angles then there are at least n cyclic points on the oval.

Theorem V.

If from a point O on a simple oval it be possible to draw n normals to S, excluding the normal at O, then there are at least $n+1$ cyclic points on the oval.

In the above theorems if O be the centre of curvature at O for any normal OP then such a normal has to be counted twice. If in addition the point P be a cyclic point then the normal OP has to be counted thrice.

Origin of the Indian Cyclic Method for the Solution of $Nx^2+1=y^2$.

BY

P. C. Sen-Gupta.

1. The object of the present paper is to discuss the probable origin of the "Cyclic Method" (*Chakrabala*) for the solution of $Nx^2+1=y^2$ in rational integers as given in Bhaskara's *Vijaganita*. Two hypotheses have been advanced as regards its origin :[1] first that the method has an ultimate Greek source and secondly that it is purely Indian. I shall first discuss the former view and shall next show that it is untenable in the light of the reasons which, I trust, are put forth herein for the first time.

2. The Cyclic Method and other Rules.

(*a*) The Cyclic Method as given by Bhaskara may be stated as follows :—

To[2] solve $Nx^2+1=y^2$, where N is a non-square integer ; start with a relation of the form :

$Na^2+k=b^2$, where a, b, k are all simple integers ; derive from it the following relation :

$$N\left(\frac{ua+b}{k}\right)^2+\frac{u^2-N}{k}=\left(\frac{bu+Na}{k}\right)$$

where and $\frac{ua+b}{k}$ are integers and u^2-N has the least value.

Repeat the operation with the new relation and obtain a fresh relation in the same form in the same way : continue to proceed in the

[1] Sir T. L. Heath's Diophantus (1910), page 281, also the references mentioned therein, viz., Tannery "Sur la mesure du cercle d'Archimède" in Mém de la soc. des sciences phys et nat. de Bordeaux, IIe Sér. IV, 1882, page 325 ; *cf.* Konen, pp. 27-28 ; Bibliotheca Mathematica, VI 3, 1905-6, pp. 271-73.

[2] Colebrooke's Indian Algebra (1817), page 175. Heath's Diophantus page 283.

same manner till *integral roots* are obtained with any of the numbers 4, 2 or 1 for the additive. Now apply "composition" [*i.e.*, the rule (b) given below] for the solution of $Nx^2+1=y^2$.

(b) If $Na^2+k=b^2$ and $Na'^2+k'=b'^2$, then will

$$N(ab'\pm a'b)^2+kk'=(Naa'\pm bb')^2.$$

This[1] lemma was first given by Brahmagupta in A.D. 628.

(c). $x=\dfrac{2\gamma}{N-\gamma^2}$, $y=\dfrac{N+\gamma^2}{N-\gamma^2}$ is a solution of $Nx^2+1=y^2$.

Neither Brahmagupta nor Bhaskara has given any formal method of deduction of the above rules. But this need not create any surprise or surmise of a Greek Origin. The first discoverers often get their results by intuition and trial.

3. Illustration of the Cyclic Method.

In the cyclic rule stated above, when a and $\dfrac{aa+b}{k}$ are both integers, it is not difficult to prove, (1) that a, k and b may be taken to be prime to one another and (2) that $\dfrac{a^2-N}{k}$ and $\dfrac{ba+Na}{k}$ are both integers. The rules (b) and (c) are also readily proved. A numerical example is given below to illustrate the cyclic rule.

To solve $67x^2+1=y^2$.

Here the relation to start with is

$$67\times 1^2-3=8^2. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

We are to solve $\dfrac{a+8}{-3}=\beta$ in integers: the suitable solution which makes a^2-N the least, is $a=7$ and $\beta=-5$, whence $\dfrac{a^2-N}{k}=6$, $\dfrac{ba+Na}{k}=-41$, *i.e.*, the relation arrived at is

$$67\times 5^2+6=41^2 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

We are now to solve $\dfrac{5a+41}{6}=\beta$ in integers: the suitable solution

[1] Colebrooke's Indian Algebra (1817), page 363 also Brahma Sphuta Siddhanta chapter XVIII, 64 and 65; Colebrooke's Indian Algebra, pages 171-172.

is $\alpha=5$ and $\beta=11$, and the new relation becomes

$$67\times 11^2-7=90^2. \quad \dots \quad \dots \quad \dots \quad (3)$$

Similarly the next relation deduced from (3) is

$$67\times 27^2-2=221^2. \quad \dots \quad \dots \quad \dots \quad (4)$$

As the additive in (4) has become -2, the cyclic method need not be followed any more; the equation is very expeditiously solved by applying the rule of Brahmagupta [§ 2 (*b*)] thus :—

We have $67\times 27^2-2=221^2$

and $67\times 27^2-2=221^2$.

$\therefore \quad 67(2\times 27\times 221)^2+4=(67\times 27^2+221^2)^2$

or $67\times 5967^2+1 \quad =48842^2$, $\quad \dots \quad \dots \quad \dots \quad (5)$

hence $x=5967$ and $y=48842$ is a solution of $67x^2+1=y^2$. From the last numerical relation repeated, application of Brahmagupta's rule leads to any number of solutions.

4. Hypothesis of ultimate Greek Origin.

(*a*) The "Diophantine Method" and the Indian Method.

M. Tannery has held that probably somewhere in one of the lost books of the Arithmetica, Diophantus solved the equation $x^2-Ay^2=1$. He has shown how from the Diophantine method, if one solution $(p,\ q)$ of $x^2-Ay^2=1$ is known a more general solution may be found :—

"Put[1] $p_1=mx-p,\ q_1=x+q$.

and suppose

$$p_1^2-Aq_1^2=m^2x^2-2mxp+p^2-Ax^2-2Aqx-Aq^2=1,$$

therefore (since $p^2-Aq^2=1$) $x=\dfrac{2mp+Aq}{m^2-A}$, and by substitution in the expression for p_1q_1, we have

$$p_1=\frac{(m^2+A)p+2Amq}{m^2-A},\ q_1=\frac{2mp+(m^2+A)q}{m^2-A}, \text{ and in fact}$$

$p_1^2-Aq_1^2=1$.

If an integral solution is wanted, one way of obtaining it is to substitute u/v for m where $u^2-Av^2=1$, *i.e.*, where $u,\ v$ is another

[1] Heath's Diophantus, page 280.

solution of the original equation, and we then have,

$p_1=(u^2+Av^2)p+2Auvq$. $q_1=2puv+(u^2+Av^2)q$.

"But this is all that we can get out of Dophantus as we have him, and it will be observed that here too *we must have ascertained two solutions of the one equation, or one solution of it and a solution of an auxiliary equation before we can apply the method.*"

It is evident from the above that there is hardly any thing common between this "Diophantine Method" and the Indian Cyclic method. It is so very imperfect that it cannot proceed without *two solutions* of the same equation. In the Indian method when *one solution* of $x^2-Ay^2=1$ is known, any number of solutions may be found by Brahmagupta's rule and that the Indian method in this case is exactly the same as the modern method. Clearly then the "ultimate Greek origin" does not lie in Diophantus's Arithmetica.

(*b*) The Archimedian approximations to a surd and the cyclic method.

Again M. Tannery's method of showing how[1] "from the Greek manner of deducing from approximation to surd a nearer approximation, it is possible by simple steps to pass to the Indian method", need not be taken as a reason for considering the indebtedness of the Indian cyclic method to any ultimate Greek origin, as there is nothing on record in the works of Archimedes which shows that he actually discovered it or applied it to the solution of $Nx^2+1=y^2$. Although I have to admit that I have not yet had access to the reference given in Dr. Heath's work, I have been able to discover a way of deriving the cyclic rule from the Archimedian method of approximating to the value of a surd, which, I trust, will not differ much from M. Tannery's. Both the Archimedian method of finding approximations to a surd and the deduction of the cyclic rule from it are exhibited below.

(1) Archimedian Method of finding $\sqrt{N}$ approximately.

Hultsch[2] proves that Archimedes "discovered and proved that

$$a\pm\frac{b}{2a}>\sqrt{a^2\pm b}>a\pm\frac{b}{2a\pm1}.$$

[1] Heath's Diophantus, page 281.

[2] Works of Archimedes (Heath, 1897), page lxxxii et sec.

Hence according to Archimedes

$\sqrt{a^2 \pm b}$ is approximately

$$= a \pm \frac{b}{2a} \text{ or } a \pm \frac{b}{2a \pm 1}.$$

As an illustration let us find the approximate values of $\sqrt{67}$.

Here $\sqrt{67} = \sqrt{8^2 + 3} = 8 + \frac{3}{2 \times 8}$, $\left(\because \sqrt{a^2 \pm b} = a \pm \frac{b}{2a}\right)$

$$= 8 + \frac{1}{5} \text{ nearly}, \quad \ldots \quad \ldots \quad (1)$$

and $\left(8\frac{1}{5}\right)^2 - 67 = \frac{6}{25}$, whence $y = 41$ and $x = 5$ is a solution of

$$y^2 - 67x^2 = 6.$$

Hence

$$\sqrt{67} = \sqrt{\left(\frac{41}{5}\right)^2 - \frac{6}{25}} = \frac{41}{5} - \frac{6/25}{2 \times \frac{41}{5} - 1}, \left(\therefore \sqrt{a^2 \mp b} = a \pm \frac{b}{2a \pm 1}\right)$$

$$= \frac{41}{5} - \frac{6}{385} = \frac{90}{11} \text{ nearly}; \quad \ldots \quad \ldots \quad (2)$$

and $\left(8 + \frac{2}{11}\right)^2 - 67 = -\frac{7}{121}$, whence $y = 90$, $x = 11$ is a solution of $y^2 - 67x^2 = -7$.

Again $\sqrt{67} = \sqrt{\left(\frac{90}{11}\right)^2 + \frac{7}{121}} = \frac{90}{11} + \frac{7/121}{2 \times \frac{90}{11} + 1}$ nearly

$$= \frac{90}{11} + \frac{1}{11 \times 27} \text{ nearly}$$

$$= \frac{221}{27} = 8 + \frac{5}{27} \quad \ldots \quad \ldots \quad (3)$$

From the last approximation it is seen that $\left(8 + \frac{5}{27}\right)^2 - 67 = \frac{2}{27^2}$ or $y = 221$, $x = 27$ is a solution of $y^2 - 67x^2 = -2$.

Thus with some difficulty we arrive at the four approximate values of $\sqrt{67}$ by the Archimedian process, *viz.*, 8, $\frac{41}{5}$, $\frac{90}{11}$ and $\frac{221}{27}$, and in passfrom one approximation to another we solve an equation of the form $y^2 - Nx^2 = k$. If however we develop $\sqrt{67}$ as a continued fraction,

the first five convergents are $\frac{8}{1}, \frac{41}{5}, \frac{90}{11}, \frac{131}{16}$ and $\frac{222}{27}$, four of which are obtained from Archimedes's rules. It is probable that he did really solve some numerical equations by his methods.

(2) Deduction of the Cyclic Method from the Archimedian process of finding $\sqrt{N}$ approximately.

From what has been shown above the Archimedian approximation proceeds either by

$$\sqrt{a^2 \pm b} = a \pm \frac{b}{2a} \text{ approximately,}$$

$$\text{or } = a \pm \frac{b}{2a \pm 1} \text{ approximately.}$$

Now when we have a relation, $Na^2+k=b^2$, $\frac{b}{a}$ is evidently the first approximation to $\sqrt{N}$. In proceding to the next approximation we have

$$N=\frac{b^2}{a^2}-\frac{k}{a^2}.$$

$$\therefore \sqrt{N}=\sqrt{\frac{b^2}{a^2}-\frac{k}{a^2}}=\frac{b}{a}-\frac{\frac{k}{a^2}}{\frac{2b}{a}-1} \text{ approximately,}$$

$$=\frac{2b^2-ab-k}{a(2b-a)}.$$

Now $(2b^2-ab-k)^2-Na^2(2b-a)^2=k(k-2ab+b^2)$
$=k\{(b-a)^2-Na^2\}$.

Let $b-a=\alpha a+\gamma$,

$\therefore\ k\{(b-a)^2-Na^2\}=k(\alpha^2a^2-Na^2)$ rejecting γ,
$=ka^2(\alpha^2-N)$.

Again $a(2b-a)=a\{b+(b-a)]=a(b+\alpha a)$ rejecting γ as before, and $2b^2-ab-k = b(b-a)+Na^2= a(b\alpha+Na)$ rejecting γ as before.

Hence we should have

$a^2(b\alpha+Na)^2-Na^2(b+\alpha a)^2= ka^2(\alpha^2-N)$, which is easily seen to be correct by actual work. Now divide both sides by k^2a^2, and we get

$$\left(\frac{b\alpha+Na}{k}\right)^2-N\left(\frac{a\alpha+b}{k}\right)^2=\frac{\alpha^2-N}{k}$$ which is exactly the cyclic rule.

This is indeed one way of arriving at the Indian rule, but the steps of putting $b-a=a\alpha+\gamma$ and rejecting γ although simple enough, are not natural. If M. Tannery has indeed deduced the cyclic rule by the above process or by a process very like the above and has imagined an "ultimate Greek" origin to the Indian cyclic rule, I feel inclined to think that he has not done sufficient justice to the Indian Algebraists. Neither Brahmagupta nor Bhaskara does anywhere connect the finding of the approximate values of $\sqrt{N}$ with the solution of $Nx^2+1=y^2$. A simpler method of arriving at the rule is given below, but this also does not seem to be the natural way of its discovery.

Let $Na^2+k=b^2$ and $Na'^2+k'=b'^2$ where a' and b' are respectively greater then a and b ; let us suppose that

$$a'=a\alpha+\gamma_1,$$

$$b'=b\alpha+\gamma_2 \text{ when } \alpha, \gamma_1, \text{ and } \gamma_2 \text{ are undetermined.}$$

$$\text{Here } b'^2-Na'^2=(b\alpha+\gamma_2)^2-N(a\alpha+\gamma_1)^2$$

$$=\alpha^2(b^2-Na^2)+2\alpha(b\gamma_2-Na\gamma_1)+(\gamma_2{}^2-N\gamma_1{}^2)$$

In order to simplify the right hand expression, assume $\gamma=b$ and $\gamma_2=Na$, so that the middle term disappears and we get

$$(b\alpha+Na)^2-N(a\alpha+b)^2=k(\alpha^2-N),$$

$$\therefore \left(\frac{b\alpha+Na}{k}\right)^2-N\left(\frac{a\alpha+b}{k}\right)^2=\frac{\alpha^2-N}{k}, \text{ which is the cyclic rule.}$$

Similarly other methods of arriving at the Indian rule may not be impossible and still not be in any way connected with its origin. We now turn to the other theory.

5. Hypothesis of a purely Indian Origin.

A most unanswerable argument for the Indianness of the method lies in its new-born naturalness and simplicity. Hankel who strongly advocates the Indian Origin of the method surmises that the Indians probably deduced it as follows :—

(a) Hankel's Method.

* Let $Na^2+k=b^2$ and $Na'^2+k'=b'^2$, then by Brahmagupta's rule we have

$$N(ab'-a'b)^2+kk'=(Naa'-bb')^2$$

Put $ab'-a'b=1$ and $Naa'-bb'=\alpha$

$$\therefore N+kk'=\alpha^2, \text{ or } k'=\frac{\alpha^2-N}{k}.$$

* Heath's Diophantus, page 284.

Next determine a' and b' from the equations

$$ab'-a'b-1=0 \quad \text{and} \quad Naa'-bb'-a=0,$$

$$\text{and } a'=\frac{aa+b}{-k} \text{ and } b'=\frac{ba+Na}{-k},$$

$$\text{or } N\left(\frac{aa+b}{k}\right)^2+\frac{(a^2-N)}{k}=\left(\frac{ba+Na}{k}\right)^2.$$

(b) Method given in M.M. Sudhakara Dvivedi's edition of Bhaskara's *Vijaganita.*

$$\text{Let } Na^2+k=b^2, \qquad \ldots \quad (1)$$

and we have identically

$$N\times 1^2+(a^2-N)=a^2. \qquad \ldots \quad (2)$$

$\therefore$ by Brahmagupta's lemma we get

$$N(aa+b)^2+k(a^2-N)=(Na+ba)^2,$$

$$\therefore\ N\left(\frac{aa+b}{k}\right)^2+\frac{a^2-N}{k}=\left(\frac{Na+ba}{k}\right)^2,$$ which is the *Chakra bala* or the Indian cyclic rule.

6. The above (b) method, as far as I have been able to ascertain seems to have been followed by all the pupils of the late M.M. Bapudev Sastri and the late Pandit Sudhakara Dvivedi. I feel inclined to believe that this elegant method is the true Indian method as transmitted through generations of *gurus.* The third rule given in §2 is easily deduced from it.

We have as before

$$N\times 1^2+(a^2-N)=a^2$$

$$\text{and } N\times 1^2+(a^2-N)=a^2,$$

$\therefore$ by the lemma of Brahmagupta, we get

$$N(2a)^2+(a^2-N)^2=(N+a^2)^2,$$

$$\text{or } N\left(\frac{2a}{a^2-N}\right)^2+1=\left(\frac{N+a^2}{a^2-N}\right)^2, \text{ whence } x=\frac{2a}{a^2-N}$$

and $y=\left(\frac{N+a^2}{a^2-N}\right)$ is a solution of $Nx^2+1=y^2$.

It is thus seen that to arrive at the Indian cyclic rule it is not at all necessary to determine an approximation to $\sqrt{N}$ either by the Archimdian method or by any other method. It is further evident that the rules are immediate deductions from the lemma of Brahmagupta, and the sole credit of finding a method for the solution of $Nx^2+1=y^2$ belongs to him.

On the motion of an ellipsoid of revolution in a viscous fluid in the light of Prof. Oseen's objection to Stokes's treatment of the case of the sphere.

BY

Bholanath Pal.

INTRODUCTION.

The motion of a sphere in a viscous fluid has been investigated by various writers including Stokes,[1] Profs. Whitehead,[2] Oseen,[3] Lamb,[4] and Burgess,[5] the results obtained being more or less satisfactory according to the degree of approximation to which the differential equations are satisfied.

In the pres nt paper, I propose (1) to obtain the solution of the problem of the motion of translation of an ellipsoid of revolution of small ellipticity in a viscous fluid, the method adopted being similar to that of Prof. Lamb for treating the corresponding problem in the case of the sphere, and (2) to show how the results obtained by me although different in some respects from those given by Oberbeck,[6] the only important writer who investigated the ellipsoidal problem before me, are free from any objection similar to that pointed out by Prof. Oseen in Stokes's treatment of the spherical problem.

In Art. 1, I reproduce the objection raised by Prof. Oseen to Stokes's solution of the spherical problem, in Arts. 2 and 3, I

[1] See his "*Scientific Papers*," Vol. 3, p. 1, or *Camb. Transactions*, Vol. 9, p. 8 (1851.)

[2] Whitehead, *Quarterly Journal of Mathematics*, Vol. 23 (1888), pp. 143-152.

[3] Oseen, *Arkiv för Mat. Astr. Och. Fysik*, Bd. 6 (1911), No. 29.

[4] Lamb, *Phil. Mag.*, series 6, Vol. 21 (1911), pp. 112-121.

[5] Burgess, *American Journ. of Math.*, Vol. 38 (1916), pp. 81-96.

[6] Oberbeck, "Ueber stationäre Flüssigkeitsbewegungen mit Berücks-ichtigung der inneren Reibung," *Crelle's Journal*, Bd. 81 (1876), pp. 62-80.

investigate the motion of the ellipsoid of revolution, in Art. 4, I compare my results with those of Oberbeck, in Art 5, I find out the resistance experienced by the ellipsoid.

I should like to express my indebtedness to Dr. Ganes Prasad at whose suggestion I took up, and under whom I carried on the investigation.

Professor Oseen's objection to Stokes's treatment for the case of a sphere.

1. The formula of Stokes for the resistance which a sphere experiences when it moves with constant and infinitely small velocity in a viscous incompressible fluid was proved by its author in the following manner.

The differential equations of Navier for the motion of the fluid, referred to a system of co-ordinates which has its origin in the centre of the sphere and which moves with that of the sphere, are

$$\left.\begin{aligned}
\rho\left\{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right\}&=-\frac{\partial p}{\partial x}+\mu\nabla^2 u,\\
\rho\left\{\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right\}&=-\frac{\partial p}{\partial y}+\mu\nabla^2 v,\\
\rho\left\{\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right\}&=-\frac{\partial p}{\partial z}+\mu\nabla^2 w,\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}&=0.
\end{aligned}\right\} \quad (1)$$

The corresponding auxiliary conditions are, for $R=\sqrt{x^2+y^2+z^2}=\infty$, $u=-U$, $v=0$, $w=0$; for $R=a$, $u=0$, $v=0$, $w=0$, if U be the velocity of the sphere and a its radius and if the x-axis be identical with the direction of the motion with the sphere.

We suppose that the motion of the fluid brought about by the sphere is stationary. Thus the first members in the three first equations fall away. Further it is clear that if at all the three functions u, v, w exist which satisfy the differential equations and the auxiliary conditions and which everywhere outside the sphere are, together with their derivatives of the first two orders, finite and continuous functions

of x, y, z, then u, v, w, $\frac{\partial u}{\partial x}$, ... $\frac{\partial w}{\partial z}$ must diminish everywhere to zero as U tends to zero. It is therefore reasonable to suppose that if U is small the so-called quadratic members $\frac{\partial u}{\partial x}$ must be of higher order of smallness than the members $\frac{\partial p}{\partial x}$, ... $\nabla^2 u$, ... and that one may consequently neglect the quadratic members. If one does this the differential equations receive the comparatively small forms

$$\mu\nabla^2 u = \partial p$$

$$\mu\nabla^2 v = \frac{\partial p}{\partial y}$$

$$\mu\nabla^2 w = \frac{\partial p}{\partial z},$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

This system of differential equations with the auxiliary conditions, for $R=\infty$, $u=-U$, $v=0$, $w=0$; for $R=a$, $u=0$, $v=0$, $w=0$, is very easy to solve. One finds that the functions

$$\left.\begin{aligned}
u &= \frac{3}{4}\frac{aU}{R^3}\left(1-\frac{a^2}{R^2}\right)x^2 - U\left(1-\frac{3}{4}\frac{a}{R}-\frac{1}{4}\frac{a^3}{R^3}\right),\\
v &= \frac{3}{4}\frac{aU}{R^3}\left(1-\frac{a^2}{R^2}\right)xy,\\
w &= \frac{3}{4}\frac{aU}{R^3}\left(1-\frac{a^2}{R^2}\right)xz,\\
p &= \frac{3}{2}\mu\frac{aU}{R^3}x
\end{aligned}\right\} \quad (2)$$

satisfy the differential equations as well as the auxiliary conditions. From these formulae one deduces easily Stokes's expression for the resistance of the sphere.

From these one finds that the value of $u\dfrac{\partial u}{\partial x}$ is

$$-\frac{3}{4}\frac{aU^2x}{R^3}\left(1-\frac{3x^2}{R^2}\right)+ \qquad \text{etc.}$$

and

$$\mu\nabla^2 u=\frac{\partial p}{\partial x}=\frac{3}{2}\mu\frac{aU}{R^3}\left(1-\frac{3x^2}{R^2}\right).$$

The ratio of the former to the latter contains x as a factor and therefore its value becomes infinite at a great distance from the sphere. For this reason the formulae in question cannot be regarded as valid at points distant from the sphere.

The motion of the ellipsoid of revolution by Lamb's method.

2. Let the equation of the spheroid be taked in the form $r=a'\{1+\epsilon P_2(\cos\theta)\}$, where ϵ is very small, so that its square and higher powers may be neglected.

Let $\bar{u}, v, w$ be the three velocity-components and suppose $\bar{u}=u+U$, so that

$$u=0, \quad v=0,\ w=0 \qquad \text{at infinity,}$$

and

$$u=-U,\ v=0,\ w=0 \qquad \text{on the surface.}$$

The hydrodynamical equations accordingly take the forms

$$\left.\begin{aligned} U\frac{\partial u}{\partial x}&=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u,\\ U\frac{\partial v}{\partial x}&=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v,\\ U\frac{\partial w}{\partial x}&=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w,\end{aligned}\right\} \qquad \dots \quad (1)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \dots \quad (2)$$

The inertia terms are to some extent taken into account.

The motion is supposed to be steady and the presence of any extraneous force is neglected ; thus from the dynamical equations we can easily obtain

$$\nabla^2 p=0, \quad \ldots \quad (3)$$

where p is the hydrodynamical pressure.

Thus we can take

$$p=\rho U \frac{\partial \phi}{\partial x} \quad \ldots \quad (3)$$

where ϕ satisfies

$$\nabla^2 \phi=0. \quad \ldots \quad (4)$$

Let us take

$$\left.\begin{aligned} u&=-\frac{\partial \phi}{\partial x}+u', \\ v&=-\frac{\partial \phi}{\partial y}+v', \\ w&=-\frac{\partial \phi}{\partial z}+w'. \end{aligned}\right\} \quad \ldots \quad (6)$$

Then from the differential equation

$$U\frac{\partial u}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u,$$

we have

$$\left(\nabla^2-\frac{U}{\nu}\frac{\partial}{\partial x}\right)u'=0.$$

Putting $k=\frac{U}{2\nu}$, this assumes the from

$$\left.\begin{aligned} &\left(\nabla^2-2k\frac{\partial}{\partial x}\right)u'=0. \\ \text{Similarly} \quad &\left(\nabla^2-2k\frac{\partial}{\partial x}\right)v'=0, \\ &\left(\nabla^2-2k\frac{\partial}{\partial x}\right)w'=0, \\ \text{and} \quad &\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0. \end{aligned}\right\} \quad \ldots \quad (7)$$

The spheroid is taken to be ovary, so that its sections perpendicular to the x—axis will be circles and therefore the vortex lines will be circles having the axis of x as a common axis. We may assume

$$\xi=0,\quad \eta=-\frac{\partial X}{\partial z},\quad \zeta=\frac{\partial X}{\partial y}, \qquad \ldots \quad (8)$$

where X is a function of x and ρ (the distance from the axis of x) only and ξ, η, ζ are the components of vorticity.

Then we must have

$$\left(\nabla^2-2k\frac{\partial}{\partial x}\right)X=0, \qquad \ldots \quad (9)$$

Thus

$$\left.\begin{aligned}
2k\frac{\partial u'}{\partial x} &= \nabla^2 u'=\frac{\partial \eta}{\partial z}-\frac{\partial \zeta}{\partial y}=-\left(\frac{\partial^2 X}{\partial y^2}+\frac{\partial^2 X}{\partial z^2}\right)\\
&=\frac{\partial^2 X}{\partial x^2}-2k\frac{\partial X}{\partial x},\\
2k\frac{\partial v'}{\partial x} &= \nabla^2 v'=\frac{\partial \zeta}{\partial x}-\frac{\partial \xi}{\partial z}=\frac{\partial^2 X}{\partial x\,\partial y},\\
2k\frac{\partial w'}{\partial x} &= \nabla^2 w'=\frac{\partial \xi}{\partial y}-\frac{\partial \eta}{\partial x}=\frac{\partial^2 X}{\partial x\,\partial z}.
\end{aligned}\right\} \qquad \ldots \quad (10)$$

Therefore

$$\left.\begin{aligned}
u' &= \frac{1}{2k}\frac{\partial X}{\partial x}-X,\\
v' &= \frac{1}{2k}\frac{\partial X}{\partial y},\\
w' &= \frac{1}{2k}\frac{\partial X}{\partial z}.
\end{aligned}\right\} \qquad \ldots \quad (11)$$

In (9), putting $X=e^{kx}X'$, we get

$$(\nabla^2-k^2)X'=0.$$

Therefore equation (9) can be written in the from

$$(\nabla^2-k^2)e^{-kx}X=0. \qquad \text{... (12)}$$

The solution of this differential equation is

$$e^{-kx}X=C\frac{e^{-kr}}{r} \qquad \text{... (13)}$$

where C is a constant[1]

From these we have finally

$$\left.\begin{aligned} u&=-\frac{\partial\phi}{\partial x}+\frac{1}{2k}\frac{\partial X}{\partial x}-X,\\ v&=-\frac{\partial\phi}{\partial y}+\frac{1}{2k}\frac{\partial X}{\partial y},\\ w&=-\frac{\partial\phi}{\partial z}+\frac{1}{2k}\frac{\partial X}{\partial z}.\end{aligned}\right\} \qquad \text{... (14)}$$

where

$$X=C\frac{e^{-k(r-x)}}{r}. \qquad \text{... (15)}$$

We have $u=0$, $v=0$, $w=0$ at infinity. Therefore ϕ must obviously involve only zonal harmonics of negative degrees and we write

$$\phi=\frac{A_0}{r}+A_1\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+A_2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)$$

$$+A_3\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)+ \qquad \text{... (16)}$$

where A_0, A_1, A_2, A_3, ... are different constants.

If we take kr very small, we have

$$X=C\left\{\frac{1}{r}-k+\frac{kx}{r}+\frac{k^2x^2}{2!}\frac{1}{r}+\ \dots\right\} \qquad \text{... (17)}$$

[1] See Prof. Lamb's *Hydrodynamics*, § 289, (Third edition).

Thus

$$\left.\begin{aligned}
\frac{1}{2k}\frac{\partial X}{\partial x}-X &= -\frac{C}{2k}\left\{\frac{4}{3}\frac{k}{r}-\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right. \\
&\qquad \left.+\frac{1}{3}kr^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\ \dots\right\}, \\
\frac{1}{2k}\frac{\partial X}{\partial y} &= -\frac{C}{2k}\left\{\qquad -\frac{\partial}{\partial y}\left(\frac{1}{r}\right)\right. \\
&\qquad \left.+\frac{1}{3}kr^2\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{r}\right)+\ \dots\right\}, \\
\frac{1}{2k}\frac{\partial X}{\partial z} &= -\frac{C}{2k}\left\{\qquad -\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right. \\
&\qquad \left.+\frac{1}{3}kr^2\frac{\partial^2}{\partial x\,\partial z}\left(\frac{1}{r}\right)+\ \dots\right\},
\end{aligned}\right\}\ \dots\ (18)$$

Therefore

$$u=-\frac{\partial}{\partial x}\left[\frac{A_0}{r}+A_1\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+A_2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\ \dots\right]$$

$$-\frac{C}{2k}\left[\frac{4}{3}\ \frac{k}{r}-\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\frac{1}{3}\ kr^2\ \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\ \dots\right], \quad \dots\ (19)$$

and similar expressions for v and w.

3. The constants C, A_0, A_1, A_2, A_3,.......can be determined from boundary condition *i.e.*, $u=-U$, for $r=a'[1+\epsilon\,P_2(\cos\theta)]$.

Therefore making use of the formula

$$P_n(\cos\theta)=(-1)^n\,\frac{r^{n+1}}{n!}\,\frac{\partial^n}{\partial x^n}\left(\frac{1}{r}\right),$$

and retaining only the first power of ϵ we have

$$-U=\frac{A_0}{a'^2}\,P_1\,(1-2\,\epsilon P_2)-\frac{A_1}{a'^3}\,2!\,P_2(1-3\epsilon\,P_2)$$

$$+\frac{A_2}{a'^4}\,3!\,P_3\,(1-4\epsilon P_2)-\frac{A_3}{a'^5}\,4!\,P_4(1-5\epsilon P_2)$$

$$+\frac{A_4}{a'^6}\,5!P_5(1-6\epsilon P_2)-\frac{A_5}{a'^7}\,6!P_6(1-7\epsilon P_2)$$

$$+\frac{A_6}{a'^8}\,7!P_7\,(-8\epsilon P_2)-\frac{A_7}{a'^9}\,-8!P_8(1-9\epsilon P_2)$$

$$+\,..$$

$$-\frac{2C}{3a'}\left(1-\epsilon P_2\right)-\frac{C}{2ka'^2}\,P_1(1-2\epsilon P_2)$$

$$-\frac{C}{3a'}\,P_2(1-\epsilon P_2)-\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (20)$$

In the above equation using the formula

$$P_2P_n=B_{n+2}\,P_{n+2}+B_n\,P_n+B_{n+2}\,P_{n+2}$$

where

$$B_{n+2}=\frac{3}{2}\,\frac{(n+1)(n+2)}{(2n+1)(2n+3)},$$

$$B_n=\frac{n(n+1)}{(2n+1)(2n+3)},$$

$$B_{n-2}=\frac{3}{2}\,\frac{n(n-1)}{(2n+1)(2n-1)},$$

and comparing the co-efficients of P_0, P_1, P_2,...the following set of equations are obtained to determine the constants .—

$$(i)\quad U+\frac{6\epsilon}{5a'^3}A_1-C\left(\frac{10-\epsilon}{15a'}\right)=0,$$

$$(ii)\quad \left(\frac{5-4\epsilon}{5a'^2}\right)A_0-\frac{4!\cdot 6\epsilon}{35a'^4}\,A_2-C\left(\frac{5-4\epsilon}{10ka'^2}\right)=0.$$

$$(iii)\quad \frac{2}{7a'^3}(7-6\epsilon)A_1-\frac{5!\cdot 2\epsilon}{7a'^5}\,A_3+C\left(\frac{7-16\epsilon}{21a'}\right)=0.$$

$$(iv)\quad \frac{6\epsilon}{5a'^2}\,A_0-\frac{2}{a'^4}\left(3-\frac{16\epsilon}{5}\right)A_2+\frac{6!\cdot 10\epsilon}{33a'^6}\,A_4-\frac{3\epsilon}{5ka'^2}\,C=0,$$

$$(v)\quad \frac{108\epsilon}{35a'^3}\,A_1-\frac{24}{a'^5}\left(1-\frac{100}{77}\,\epsilon\right)A_3+\frac{7!.45\epsilon}{143a'^7}A_5+\frac{6\epsilon}{35a'}\,C=0,$$

$$(iv)\quad \frac{80\epsilon}{7a'^4}A_2-\frac{120}{a'^6}\left(1-\frac{20}{13}\epsilon\right)A_4+\frac{8!.21\epsilon}{65a'^8}\,A_6=0,$$

$$(vii)\quad \frac{600\epsilon}{11a'^5}\,A_3-\frac{144}{a'^7}\left(5-\frac{98}{11}\epsilon\right)A_5+\frac{9!.21\epsilon}{85a'^9}\,A_7=0, \qquad \ldots \quad (21)$$

etc.

We have taken ϵ to be very small and retained only its first power; so we put for A_1 in (i) its value for the case of the sphere, which is $-\dfrac{Ua'^3}{4}$.

Thus from (i) we have

$$C=\frac{3Ua'}{2}\left(1-\frac{1}{5}\epsilon\right) \qquad \ldots \qquad \ldots \quad (22)$$

We have A_2, A_3,......all zero for the case of the sphere, therefore for the case of the spheroid the values of A_2, A_3,......will contain ϵ as a factor; therefore we neglect all terms like ϵA_2, ϵA_3,....as we take into account only the first power of ϵ.

Thus from (ii) we have

$$A_0=\frac{3va'}{2}\left(1-\frac{1}{5}\epsilon\right). \qquad \ldots \qquad \ldots \quad (23)$$

Similarly from (iii) we get

$$A_1=-\frac{Ua'^3}{4}\left(1-\frac{57}{35}\epsilon\right). \qquad \ldots \qquad \ldots \quad (24)$$

From (vi), we get

$$A_2=0.$$

Therefore from the equations it is clear that all the constants such as A_4, A_6.......(*i.e.* the A's with even suffixes) are all zero.

From (v) we have

$$A_3=-\frac{3Ua'^5}{140}\epsilon. \qquad \ldots \qquad \ldots \quad (25)$$

A_5, A_7,...will contain ϵ^2, ϵ^3,...as factors and consequently they are neglected.

Thus finally after a slight simplification we find

$$u=\frac{Ua'}{4}\left[-4\left(1-\tfrac{1}{5}\epsilon\right)\frac{1}{r}-r^2\left(1-\tfrac{1}{5}\epsilon\right)\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)\right.$$
$$\left.+a'^2\left(1-\frac{57}{35}\epsilon\right)\frac{\partial^2}{\partial x^2}\left(\frac{3}{r}\right)+\frac{3}{35}a'^4\epsilon\,\frac{\partial^4}{\partial x^4}\left(\frac{1}{r}\right)\right],$$

$$v=\frac{Ua'}{4}\left[-r^2\left(1-\tfrac{1}{5}\epsilon\right)\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{r}\right)+a'^2\left(1-\frac{57}{35}\epsilon\right)\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{r}\right)\right.$$
$$\left.+\frac{3}{35}a'^4\epsilon\,\frac{\partial^4}{\partial x^3\,\partial y}\left(\frac{1}{r}\right)\right],$$

$$w=\frac{Ua'}{4}\left[-r^2\left(1-\tfrac{1}{5}\epsilon\right)\frac{\partial^2}{\partial x\,\partial z}\left(\frac{1}{r}\right)+a'^2\left(1-\frac{57}{35}\epsilon\right)\frac{\partial^2}{\partial x\,\partial z}\left(\frac{1}{r}\right)\right.$$
$$\left.+\frac{3}{35}a'^4\epsilon\,\frac{\partial^4}{\partial x^3\,\partial z}\left(\frac{1}{r}\right)\right].$$

etc.

Oberbeck's solution.

4. If an ellipsoid moves in an infinite mass of viscous liquid with the general velocity U, parallel to the axis of x such that $u=0$, $v=0$, $w=0$ at the surface and $u=\mathrm{U}$, $v=0$, $w=0$ at infinity, then Oberbeck has found out the following results for the motion of the liquid

$$u=\mathrm{U}+\lambda'\left[x\frac{\partial \mathrm{Q}}{\partial x}-\mathrm{Q}+\mu'\frac{\partial^2 \mathrm{P}}{\partial x^2}\right],$$

$$v=\lambda'\left[x\frac{\partial \mathrm{Q}}{\partial y}+\mu'\frac{\partial^2 \mathrm{P}}{\partial x\,\partial y}\right],$$

$$w=\lambda'\left[x\frac{\partial \mathrm{Q}}{\partial z}+\mu'\frac{\partial^2 \mathrm{P}}{\partial x\,\partial z}\right].$$

where

$$\mathrm{P}=-\pi abc\int_\lambda^\infty \frac{\dfrac{x^2}{a^2+s}+\dfrac{y^2}{b^2+s}+\dfrac{z^2}{c^2+s}-1}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}}\,ds,$$

and

$$\mathrm{Q}=2\pi abc\int_\lambda^\infty \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}},$$

λ being the positive root of the equation

$$\frac{x^2}{a^2+s}+\frac{y^2}{b^2+s}+\frac{z^2}{c^2+s}=1,$$

and λ' and μ' are constants such that

$$\lambda'=\frac{\mathrm{U}}{\mathrm{Q}_0+a^2\mathrm{A}_0},\ \text{and}\ \mu'=a^2,$$

where

$$\mathrm{Q}_0=2\pi abc\int_0^\infty \frac{ds}{\sqrt{(a^2+s)\,(b^2+s)\,(c^2+s)}},$$

and

$$\mathrm{A}_0=2\pi abc\int_0^\infty \frac{ds}{(a^2+s)\ \sqrt{(a_2+s)\,(b^2+s)\,(c^2+s)}}.$$

I propose now to deduce, from these results given by Oberbeck, the values of u, v, w for the case, when $b=c$.

I take the ellipsoid of revolution to be ovary (*i.e.*, $b=c$) and its equation to be of the forms $r=a'[1+\epsilon P_2(\cos\theta)]$ where ϵ is very small, so that we may neglect its square and higher powers.

Therefore

$$a=a'\,(1+\epsilon),\ b=a'\,(1-\frac{\epsilon}{2}),\ a^2-b^2=a^2e^2,\ \text{and } \epsilon=\frac{e^2}{3}.$$

We have

P*=the potential due to the spheroid at an external point

$$=\frac{3M}{(a^2-b^2)^{\frac{1}{2}}}\left[\frac{1}{1\cdot 3}\frac{(a^2-b^2)^{\frac{1}{2}}}{r}+\frac{1}{3\cdot 5}\frac{(a^2-b^2)^{\frac{3}{2}}}{r^3}P_2(\cos\theta)+\ \ldots\right],$$

where $$M=\frac{4\pi ab^2}{3},\ \text{and } \cos\theta=\frac{x}{r}.$$

Similarly

$$Q^*=-\frac{M'}{(a^2-b^2)^{\frac{1}{2}}}\left[\frac{(a^2-b^2)^{2}}{r}+\frac{1}{3}\frac{(a^2-b^2)^{\frac{3}{2}}}{r^3}P_2(\cos\theta)+\ \ldots\right],$$

where $$M'=4\pi ab^2.$$

Therefore

$$P=\frac{4\pi ab^2}{3}\cdot\frac{1}{r}+\frac{4\pi ab^2}{3}\frac{(a^2-b^2)}{2\cdot 5}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\ \ldots\ \ldots,$$

and

$$Q=\frac{4\pi ab^2}{3}\cdot\frac{3}{r}+\frac{4\pi ab^2}{3}\frac{(a^2-b^2)}{2}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+\ \ldots\ \ldots.$$

Substituting these values of P and Q, we have

$$u=U+\frac{U}{Q_0+a^2A_0}\frac{4\pi ab^2}{3}\left[3x\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\frac{a^2e^2}{2}x\frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right)\right.$$

$$-\frac{3}{r}-\frac{a^2e^2}{2}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)+a^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)$$

$$\left.+\frac{a^4e^2}{10}\frac{\partial^4}{\partial x^4}\left(\frac{1}{r}\right)\right],$$

neglecting higher powers of e^2.

* See Byerly's, *Spherical Harmonics*, pp. 155 and 156.

We have

$$x=\frac{P_1}{r}\,,\ P_1{}^2=\frac{P_0+2P_2}{3},\ \text{and}\ P_1P_3=\frac{4P_4+3P_2}{7}\,.$$

Therefore

$$u=U+\frac{U}{Q_0+a^2A_0}\,\frac{4\pi ab^2}{3}\left[-\frac{4P_0}{r}-\frac{2P_2}{r}+a^2\left(1-\frac{8}{7}e^2\right)\frac{2P_2}{r^3}\right.$$
$$\left.-\frac{12}{7}a^2e^2\,\frac{P_4}{r^3}+\frac{4\,!}{10}\,a^4e^2\,\frac{P_4}{r^5}\right].$$

i.e.,

$$u=U+\frac{U}{Q_0+a^2A_0}\,\frac{4\pi a'^3}{3}\left[-\frac{4P_0}{r}-\frac{2P_2}{r}+a'^2\left(1-\frac{10}{7}\epsilon\right)\frac{2P_2}{r^3}\right.$$
$$\left.-\frac{3}{14}a'^2\epsilon\,\frac{4\,!\,P_4}{r^3}+\frac{3}{10}a'^4\epsilon\,\frac{4\,!\,P_4}{r^5}\right].$$

From the boundary conditions we have

$$u=0,\ \text{when}\ r=a'(1+\epsilon P_2)$$

Therefore

$$0=U+\frac{U}{Q_0+a^2A_0}\,\frac{4\pi a'^3}{3}\left[-\frac{4P_0}{a'}(1-\epsilon P_2)-\frac{2P_2}{a'}(1-\epsilon P_2)\right.$$
$$+a'^2\left(1-\frac{10}{7}\epsilon\right)\frac{2P_2}{a'^3}(1-3\epsilon P_2)$$
$$\left.-\frac{3}{14}a'^2\epsilon\,\frac{4\,!\,P_4}{a'^3}+\frac{3}{10}a'^4\epsilon\,\frac{4\,!\,P_4}{a'^5}\right],$$

neglecting square and higher powers of ϵ.

From this equation comparing the coefficients of P_0, we have

$$U+\frac{U}{Q_0+a^2A_0}\,\frac{4\pi a'^3}{3}\left[-\frac{4}{a'}+\frac{2\epsilon}{5a'}-\frac{6\epsilon}{5a'}\right]=0,$$

since $$(P_2)^2=\frac{18}{35}P_4+\frac{2}{7}P_2+\frac{1}{7}P_0.$$

From this we find

$$Q_0+a^2A_0=\frac{4\pi a'^3}{3}\left(4+\frac{4}{5}\epsilon\right).$$

Thus finally we find

$$u=U+\frac{Ua'}{4}\left[-4\left(1-\frac{1}{5}\epsilon\right)\frac{1}{r}-r^2\left(1-\frac{1}{5}\epsilon\right)\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)\right.$$
$$+a'^2\left(1-\frac{57}{35}\epsilon\right)\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)-\frac{3}{14}a'^2\epsilon r^2\frac{\partial^4}{\partial x^4}\left(\frac{1}{r}\right)$$
$$\left.+\frac{3}{10}a'^4\epsilon\frac{\partial^4}{\partial x^4}\left(\frac{1}{r}\right)\right].$$

Similarly we can find out the values of v and w in terms of the differential coefficients of $\frac{1}{r}$, as

$$v=\frac{Ua'}{3}\left[-r^2\left(1-\frac{1}{5}\epsilon\right)\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{r}\right)+a'^2\left(1-\frac{57}{35}\epsilon\right)\frac{\partial^2}{\partial x\partial y}\left(\frac{1}{r}\right)\right.$$

$$\left.-\frac{3}{14}a'^2\epsilon r^2\frac{\partial^4}{\partial x^3\partial y}\left(\frac{1}{r}\right)+\frac{3}{10}a'^4\epsilon\frac{\partial^4}{\partial x^3\partial y}\left(\frac{1}{r}\right)\right].$$

$$w=\frac{Ua'}{4}\left[-r^2\left(1-\frac{1}{5}\epsilon\right)+a'^2\left(1-\frac{57}{35}\epsilon\right)\frac{\partial^2}{\partial x\partial z}\left(\frac{1}{r}\right)\right.$$

$$\left.-\frac{3}{14}a'^2\epsilon r^2\frac{\partial^4}{\partial x^3\partial z}\left(\frac{1}{r}\right)+\frac{3}{10}a'^4\epsilon\frac{\partial^4}{\partial x^3\partial z}\left(\frac{1}{r}\right)\right].$$

Resistance

5 Next I propose to find out an expression for the resistance experienced by the ellipsoid in moving through the liquid.

Let F denote the resistance; then for the case of the ellipsoid of three unequal axes. Oberbeck has found out that

$$\text{the resistance}=6\mu\pi RU.\ \text{and}\ R=\frac{8}{3}\,\frac{abc}{X_0+a^2\alpha_0}$$

where

$$X_0=abc\int_0^\infty\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}\ \text{and}$$

$$\alpha_0=abc\int_0^\infty\frac{d\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}$$

In the present case $b=c$, $a=a'(1+\epsilon)$, $b=a'\left(1-\frac{\epsilon}{2}\right)$

and
$$X_0+a^2\alpha_0=\frac{2}{3}a'^2\left(4+\frac{4}{5}\epsilon\right).$$

Therefore
$$R=\frac{8}{3}\,\frac{a'^3}{\frac{2}{3}a'^2\left(4+\frac{4}{5}\epsilon\right)}\qquad\frac{a'}{1+\frac{1}{5}\epsilon}$$

and consequently
$$F=6\pi\mu a'U\left(1-\frac{1}{5}\epsilon\right);$$

and from my results I get the same expression for the resistance.

6. The method used by me for the ellipsoid of revolution is capable of being extended to the case of the ellipsoid of three unequal axes.

On a class of ellipsoidal harmonics and a method of solving the wave equation in ellipsoidal coordinates.

BY

SUDHANSUKUMAR BANERJI.

1. In the present paper I have developed a class of ellipsoidal harmonics which are solutions of Laplace's equation and then have used these harmonics in solving the wave equation in ellipsoidal coordinates. The ellipsoidal coordinates used in this paper are (ρ, θ, ϕ) defined by

$$\begin{aligned} x &= a\rho \sin\theta \cos\phi, \\ y &= b\rho \sin\theta \sin\phi, \\ z &= c\rho \cos\theta. \end{aligned} \qquad \ldots \quad (1)$$

where ρ=constant obviously determines a set of similar and similarly situated ellipsoids. This system being analogous to the ordinary polar coordinates has got certain advantage over the more usual system λ, μ, ν representing a set of confocal ellipsoidal surfaces, hyperboloids of one sheet and hyperboloids of two sheets respectively but has also got certain disadvantage in as much as it does not form an orthogonal system. The ellipsoidal harmonics in the coordinates (ρ, θ, ϕ) developed in this paper will be found to be simpler and more convenient for application to physical problems than the Lame's functions. But perhaps the most remarkable application of these harmonics consists in the use that has been made of them in this paper in solving the wave equation in these coordinates. This equation which was first transformed by Mathieu[1] in λ, μ, ν was found to be so unmanageable that he had to content himself with approximating to its solution for the special case of an ellipsoid of revolution. Subsequent writers including Prof. Niven[2] have simply improved upon the approximations of Mathieu.

[1] *Cours de Physique Mathematique*, Ch. IX.

[2] *Phil. Trans.*, Vol. CLXXI, (1880).

2. It is well-known that if (r, θ, ϕ) denote the spherical polar coordinates of a point (x, y, z),

$$r^n \, P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\phi$$

$$= \frac{(n+m)(n+m-1)\dots(n+1)}{2\pi} (-1)^{\frac{m}{2}} \int_0^{2\pi} (z + ix\cos u + iy\sin u)^n \begin{matrix}\cos\\ \sin\end{matrix} mu du. \qquad \dots (2)$$

Obviously, by a generalisation of this expression we can define a function $C_n^m(\theta, \phi)$ by the expression

$$C_n^m(\theta, \phi) = \frac{(n+m)(n+m-1)\dots(n+1)}{2\pi} (-1)^{\frac{m}{2}}$$

$$\int_0^{2\pi} (c\cos\theta + ia\sin\theta\cos\phi\cos u + ib\sin\theta\sin\phi\sin u)^n \cos mu du \qquad \dots (3)$$

With this definition for the function $C_n^m(\theta, \phi)$, it is easy to see that $\rho^n C_n^m(\theta, \phi)$ is a solution of Laplace's equation in the ellipsoidal coordinates (ρ, θ, ϕ) defined by (1).

Similarly we can define a function $S_n^m(\theta, \phi)$ by the expression

$$S_n^m(\theta, \phi) = \frac{(n+m)(n+m-1)\dots(n+1)}{2\pi} (-1)^{\frac{m}{2}}$$

$$\int_0^{2\pi} (c\cos\theta + ia\sin\theta\cos\phi\cos u + ib\sin\theta\sin\phi\sin u)^n \sin mu du \qquad \dots (4)$$

and $\rho^n S_n^m(\theta, \phi)$ is another solution of Laplace's equation in $(\rho, \theta \phi)$.

The function corresponding to the Legendre's function can be defined by

$$C_n(\theta, \phi) = \frac{1}{2\pi} \int_0^{2\pi} (c\cos\theta + ia\sin\theta\cos\phi\cos u + ib\sin\theta\sin\phi\sin u)^n du$$

$$= \frac{1}{2\pi} \int_0^{2\pi} [c\cos\theta + i(a^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi)^{\frac{1}{2}} \cos u]^n du \qquad \dots (5)$$

and $\rho^n C_n(\theta, \phi)$ is a solution of Laplace's equation in (ρ, θ, ϕ). When $a=b$, the function is independent of ϕ, *i.e.*,

$$C_n(\theta)=\frac{1}{2\pi}\int_0^{2\pi}(c \cos \theta+ia \sin \theta \cos u)^n \, du.$$

3. To obtain the harmonics which vanish at infinity we define the functions

$$\mathfrak{C}_n^m(\theta, \phi)=(-1)^m \frac{n(n-1) \ldots (n-m+1)}{2\pi}$$

$$\int_0^{2\pi}\frac{\cos mu\, du}{(c \cos \theta+ia \sin \theta \cos \phi \cos u+ib \sin \theta \sin \phi \sin u)^{n+1}} \qquad (6)$$

and

$$\mathfrak{S}_n^m(\theta, \phi)=(-1)^m \frac{n(n-1) \ldots (n-m+1)}{2\pi}$$

$$\int_0^{2\pi}\frac{\sin mu\, du}{(c \cos \theta+ia \sin \theta \cos \phi \cos u+ib \sin \theta \sin \phi \sin u)^{n+1}} \quad \ldots \quad (7)$$

and obviously $\mathfrak{C}_n^m(\theta, \phi)/\rho^{n+1}$ and $\mathfrak{S}_n^m(\theta, \phi)/\rho^{n+1}$ are solutions of Laplace's equation which vanish at infinity.

4. By an application of Green's theorem we can easily prove that the functions $C_n^m(\theta,\phi)$, $S_n^m(\theta,\phi)$, $\mathfrak{C}_n^m(\theta,\phi)$ and $\mathfrak{S}_n^m(\theta,\phi)$ defined above all satisfy conjugate properties.

The element of volume in the co-ordinates (ρ, θ, ϕ) is

$$abc\, \rho^2 \sin \theta \, d\rho \, d\theta \, d\phi \quad \ldots \quad (8)$$

which can also be written in the form

$$dp.dS, \quad \ldots \quad (9)$$

where dS is a surface element and

$$dp=p_0 d\rho,$$

$$p_0=\frac{abc}{(b^2c^2 \sin^2\theta \cos^2\phi+c^2a^2 \sin^2\theta \sin^2\phi+a^2b^2 \cos^2\theta)^{\frac{1}{2}}}. \quad \ldots \quad (10)$$

So that the surface element dS can be written in the form

$$dS=\frac{abc}{p_0}\rho^2\sin\theta\,d\theta\,d\phi. \qquad \ldots \quad (11)$$

Now, by Green's theorem, if Φ and Φ' be two functions which satisfy Laplace's equation, we have

$$\iint\left(\Phi'\frac{\partial\Phi}{\partial n}-\Phi\frac{\partial\Phi'}{\partial n}\right)dS=0. \qquad \ldots \quad (12)$$

Hence since $\dfrac{\partial}{\partial n}=\dfrac{1}{p_0}\dfrac{\partial}{\partial\rho}$. we see at once that the functions $C_n^m(\theta,\phi)$ and $S_n^m(\theta,\phi)$ defined above satisfy the following conjugate properties :—

$$\int_0^\pi\int_0^{2\pi}C_n^m(\theta,\phi)C_{n'}^m(\theta,\phi)\frac{\sin\theta}{p_0^2}\,d\theta d\phi=0, \qquad (n\neq n') \ldots \quad (13)$$

$$\int_0^\pi\int_0^{2\pi}S_n^m(\theta,\phi)\,S_{n'}^m(\theta,\phi)\frac{\sin\theta}{p_0^2}\,d\theta\,d\phi=0, \qquad (n\neq n') \ldots \quad (14)$$

Also

$$\int_0^\pi\int_0^{2\pi}[C_n^m(\theta,\phi)]^2\frac{\sin\theta}{p_0^2}\,d\theta\,d\phi=\text{const.}=\lambda_n \text{ (say)}, \qquad \ldots \quad (15)$$

$$\int_0^\pi\int_0^{2\pi}[S_n^m(\theta,\phi)]^2\frac{\sin\theta}{p_0^2}\,d\theta d\phi=\text{const.}=\lambda_n. \qquad \ldots \quad (16)$$

Similarly for the functions $\mathfrak{C}_n^m(\theta,\phi)$, $\mathfrak{S}_n^m(\theta,\phi)$.

5. It is interesting to note the following relations between the functions $C_n^m(\theta,\phi)$, $S_n^m(\theta,\phi)$, $\mathfrak{C}_n^m(\theta,\phi)$ and $\mathfrak{S}_n^m(\theta,\phi)$:—

$$C_n^m(\theta,\phi)=(a^2\sin^2\theta\cos^2\phi+b^2\sin^2\theta\sin^2\phi+c^2\cos^2\theta)^{\frac{2n+1}{2}}\,\mathfrak{C}_n^m(\theta,\phi), \qquad (17)$$

$$S_n^m(\theta,\phi)=(a^2\sin^2\theta\cos^2\phi+b^2\sin^2\theta\sin^2\phi+c^2\cos^2\theta)^{\frac{2n+1}{2}}\,\mathfrak{S}_n^m(\theta,\phi). \qquad (18)$$

When $a=b=c=1$, $C_n^m(\theta,\phi)=\mathfrak{C}_n^m(\theta,\phi)=P_n^m(\cos\theta)\cos m\phi$

and $S_n^m(\theta,\phi)=\mathfrak{S}_n^m(\theta,\phi)=P_n^m(\cos\theta)\sin m\phi$.

The values of the functions $C_n^m(\theta,\phi)$, etc., can be expressed in terms of the hypergeometric function.

Thus we get

$$C_n^m(\theta,\phi)=\frac{(n+m)!}{(n-m)!}\,\frac{i^m R^{2n+1} r^m \tan^m\theta}{2^m m!\,(c\cos\theta)^{n+1}}\,F\left(\frac{n+m+1}{2},\frac{n+m+2}{2},m+1,\right.$$
$$\left.-r^2\tan^2\theta\right)\cos m\psi,$$

$$S_n^m(\theta,\phi)=\frac{(n+m)!}{(n-m)!}\,\frac{i^m R^{2n+1} r^m \tan^m\theta}{2^m m!\,(c\cos\theta)^{n+1}}\,F\left(\frac{n+m+1}{2},\frac{n+m+2}{2},m+1,\right.$$
$$\left.-r^2\tan^2\theta\right)\sin m\psi.$$

where F is a hypergeometric function of the four elements within the parenthesis and

$$R^2=a^2\sin^2\theta\cos^2\phi+b^2\sin^2\theta\sin^2\phi+c^2\cos^2\theta,$$

$$r^2=(a^2\cos^2\phi+b^2\sin^2\phi)/c^2,$$

$$\tan\psi=b/a\tan\phi,$$

c being the greatest axis of the ellipsoid.

5. It is well-known that

$$P_n(\sin\theta\cos\phi\sin u\cos v+\sin\theta\sin\phi\sin u\sin v+\cos\theta\cos u)$$

$$=P_n(\cos\theta)P_n(\cos u)+2\sum_{m=1}^{m=n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos u)\cos m(\phi-v)$$

If we write

$$x=r\sin\theta\cos\phi, \qquad x'=r'\sin u\cos v,$$

$$y=r\sin\theta\sin\phi, \qquad y'=r'\sin u\sin v,$$

$$z=r\cos\theta, \qquad z'=r'\cos u,$$

and $$\Pi_{n,m}=\frac{(n+m)(n+m-1)\ldots(n+1)}{2\pi}(-1)^{\frac{m}{2}},$$

then we can write the above identity in the form

$$r^n r'^n\,P_n(\sin\theta\cos\phi\sin u\cos v+\sin\theta\sin\phi\sin u\sin v+\cos\theta\cos u)$$

$$=\frac{1}{4\pi^2}\int_0^{2\pi}(z+ix\cos\omega+iy\sin\omega)^n d\omega\times\int_0^{2\pi}(z'+ix'\cos\omega+iy'\sin\omega)^n d\omega$$

$$+2\sum_{m=1}^{m=n}\frac{(n-m)!}{(n+m)!}\Pi^2_{n,m}\Bigg[\int_0^{2\pi}(z+ix\cos\omega+iy\sin\omega)^n\cos m\omega d\omega$$

$$\times\int_0^{2\pi}(z'+ix'\cos\omega+iy'\sin\omega)^n\cos m\omega d\omega$$

$$+\int_0^{2\pi}(z+ix\cos\omega+iy\sin\omega)^n\sin m\omega d\omega$$

$$\times\int_0^{2\pi}(z'+ix'\cos\omega+iy'\sin\omega)^n\sin m\omega d\omega\Bigg],$$

that is to say in the form

$$\frac{1}{2\pi}\int_0^{2\pi}[xx'+yy'+zz'+i\{(yz'-zy')^2+(zx'-xz')^2+(xy'-yx')^2\}^{\frac{1}{2}}\cos\omega]^n d\omega$$

$$=\frac{1}{4\pi^2}\int_0^{2\pi}(z+ix\cos\omega+iy\sin\omega)^n d\omega\times\int_0^{2\pi}(z'+ix'\cos\omega+iy'\sin\omega)^n d\omega$$

$$+\sum_{m=1}^{=n}\frac{(n-m)!}{(n+m)!}\Pi^2_{n,m}\Bigg[\int_0^{2\pi}(z+ix\cos\omega+iy\sin\omega)^n\cos m\omega d\omega$$

$$\times\int^{2\pi}(z'+ix'\cos\omega+iy'\sin\omega)^n\cos m\omega d\omega$$

$$+\int^{2\pi}(z'+ix'\cos\omega+iy'\sin\omega)^n\sin m\omega d\omega$$

$$\times\int_0^{2\pi}(z'+ix'\cos\omega+iy'\sin\omega)^n\sin m\omega d\omega\Bigg].$$

Now if we write

$$x=a\rho\sin\theta\cos\phi,\qquad x'=r'\sin u\cos v,$$

$$y=b\rho\sin\theta\sin\phi,\qquad y'=r'\sin u\sin v,$$

$$z=c\rho\cos\theta,\qquad z'=r'\cos u,$$

we get

$$\frac{1}{2\pi}\int_0^{2\pi}\Big[a\sin\theta\cos\phi\sin u\cos v+b\sin\theta\sin\phi\sin u\sin v+c\cos\theta\cos u$$

$$+i\{(b\sin\theta\sin\phi\cos u-c\sin u\sin v\cos\theta)^2$$

$$+(c\sin u\cos v\cos\theta-a\sin\theta\cos\phi\cos u)^2$$

$$+(a\sin\theta\cos\phi\sin u\sin v-b\sin\theta\sin\phi\sin u\cos v)^2\}^{\frac{1}{2}}\cos\omega\Big]^n d\omega$$

$$=\frac{1}{4\pi^2}\int_0^{2\pi}(c\cos\theta+ia\sin\theta\cos\phi\cos\omega+ib\sin\theta\sin\phi\sin\omega)^n\,d\omega$$

$$\times\int_0^{2\pi}(\cos u+i\sin u\cos v\cos\omega+i\sin u\sin v\sin\omega)^n\,d\omega$$

$$+2\sum_{m=1}^{m=n}\frac{(n-m)!}{(n+m)!}\,\Pi^2_{n,\,m}\Bigg[\int_0^{2\pi}(c\cos\theta+ia\sin\theta\cos\phi\cos\omega+ib\sin\theta\sin\phi\sin\omega)^n\cos m\omega d\omega$$

$$\times\int_0^{2\pi}(\cos u+i\sin u\cos v\cos\omega+i\sin u\sin v\sin\omega)^n\cos m\omega d\omega$$

$$+\int_0^{2\pi}(c\cos\theta+ia\sin\theta\cos\phi\cos\omega+ib\sin\theta\sin\phi\sin\omega)^n\sin m\omega d\omega$$

$$\times\int_0^{2\pi}(\cos u+i\sin u\cos v\cos\omega+i\sin u\sin v\sin\omega)^n\sin m\omega d\omega\Bigg].$$

In other words, we get in accordance to our previous definition

$$C_n(\theta,\phi\,;u,v)=C_n(\theta,\phi)\,P_n(\cos u)$$

$$+2\sum_{m=1}^{m=n}\frac{(n-m)!}{(n+m)!}\Big[C_n^m(\theta,\phi)\,P_n^m(\cos u)\cos mv+S_n^m(\theta,\phi)\,P_n^m(\cos u)\sin mv\Big]\quad\ldots\quad(19)$$

7. We shall now obtain the solution of the wave equation

$$(\nabla^2+k^2)V=0\qquad\ldots\quad(20)$$

in terms of the functions introduced in the previous articles.

It is well-known that

$$V=\int_0^{\pi}\int_0^{2\pi} e^{ik(x\sin u\cos v+y\sin u\sin v+z\cos u)} f(u,v)\,du\,dv \quad \ldots \quad (21)$$

represents a solution of the above equation in cartesian coordinates.

Hence in the coordinates (ρ, θ, ϕ), a solution of the above equation will be given by

$$V=\int_0^{\pi}\int_0^{2\pi} e^{ik\rho(a\sin\theta\cos\phi\sin u\cos v+b\sin\theta\sin\phi\sin u\sin v+c\cos\theta\cos u)} f(u,v)\,du\,dv \quad \ldots \quad (22)$$

It is obvious that by properly choosing the function $f(u, v)$ we can construct a set of solutions of the equation $(\nabla^2+k^2)V=0$ in ρ, θ, ϕ.

Let us define the function $\psi_n(k\rho)$ by the relation

$$\psi_n(k\rho)=\frac{i^{-n}}{2\pi}\int_0^{\pi}\int_0^{2\pi} e^{ik\rho c\cos\theta}\, C_n(\theta,\phi)\,\frac{\sin\theta}{p_0^{\,2}}\,d\theta\, d\phi. \quad \ldots \quad (23)$$

When $a=b$,

$$\psi_n(k\rho)=\frac{i^{-n}}{b^2c^2}\int_0^{\pi} e^{ik\rho c\cos\theta}\, C_n(\theta)(c^2\sin^2\theta+a^2\cos^2\theta)\sin\theta\, d\theta \quad \ldots \quad (24)$$

By expanding $e^{ik\rho c\cos\theta}$ in the exponential series, it is easy to evaluate the integral term by term and to obtain an expression for $\psi_n(k\rho)$ in a series of ascending powers of $k\rho$.

One way[1] of expressing the result is

$$\psi_n(k\rho)=\sum_s A_s(ck\rho)^s,$$

[1] See a note by Prof. Baker on a formula connected with the theory of spherical harmonics, *Proc. Lond. Math. Soc.*, Vol. XV, (1916).

where the summation extends for all values of s for which $n+s+2$ is an even integer and

$$A_s = \frac{i^{s-n}}{4\pi^2 a^2 b^2 c^2}\left[\frac{2^s \partial^s}{\partial \zeta^s} U_s + 2^{s-2}\frac{s(s-1)}{1!}\frac{\partial^{s-2}}{\partial \zeta^{s-2}} U_{s-2}\right.$$

$$\left. + 2^{s-4}\frac{s(s-1)(s-2)(s-3)}{2!}\frac{\partial^{s-4}}{\partial \zeta^{s-4}} U_{s-4} + \dots\right],$$

$$U_q = \frac{1}{[\frac{1}{2}(n+2-q)]!}\left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2}\right]^{\frac{1}{2}(n+2-q)} U,$$

$$U = (b^2c^2\xi^2 + c^2a^2\eta^2 + a^2b^2\zeta^2)\int_0^{2\pi}(c\zeta + ia\xi\cos\omega + ib\eta\sin\omega)^n d\omega$$

and ξ, η, ζ stand for $\sin\theta\cos\phi$, $\sin\theta\sin\phi$ and $\cos\theta$.

It can be shown that the differential equation $(\nabla^2 + k^2)\,V = 0$, can be transformed to the form

$$\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi_n}{\partial\rho}\right)\frac{C_n{}^m(\theta,\phi)}{\rho_n{}^2} + \frac{1}{\rho}\frac{\partial\psi_n}{\partial\rho}\left[2nD_n{}^m(\theta,\phi)\right.$$

$$\left. + \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)C_n{}^m(\theta,\phi)\right] + k^2\psi_n C_n{}^m(\theta,\phi) = 0.$$

where
$$D_n{}^m(\theta,\phi) = \frac{(n+m)(n+m-1)\dots(n+1)}{2\pi}(-1)^{\frac{m}{2}}$$

$$\times\int_0^{2\pi}(c\cos\theta + ia\sin\theta\cos\phi\cos u + ib\sin\theta\sin\phi\sin u)^{n-1}$$

$$\left(\frac{\cos\theta}{c} + \frac{i\sin\theta\cos\phi\cos u}{a} + \frac{i\sin\theta\sin\phi\sin u}{b}\right)\cos mu\,du$$

and that $\psi_n(k\rho)$ satisfies the equation

$$\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi_n}{\partial\rho}\right) + \alpha_n\frac{1}{\rho}\frac{\partial\psi_n}{\partial\rho} + \beta_n k^2\psi_n = 0,$$

where $$\alpha_n = \frac{1}{\lambda_n}\left[2n\int_0^{\pi}\int_0^{2\pi} D_n^m(\theta, \phi)C_n^m(\theta, \phi) \sin \theta d\theta d\phi\right.$$

$$\left.\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\int_0\int_0^{2\pi}[C_n^m(\theta, \phi)]^2 \sin \theta d\theta d\phi\right]$$

$$\beta_n = \frac{1}{\lambda_n}\int_0^{\pi}\int_0^{2\pi} [C_n^m(\theta, \phi)]^2 \sin \theta d\theta d\phi.$$

With this definition for $\psi_n(k\rho)$, it is obvious that we can expand $e^{ik\rho c \cos \theta}$ in a series of the type

$$e^{ik\rho c \cos \theta} = A_0\psi_0(k\rho)+A_1\psi_1(k\rho)C_1(\theta, \phi)+A_2\psi_2(k\rho)C_2(\theta, \phi)+\ldots$$
$$+A_n\psi_n(k\rho)C_n(\theta,\phi)+\ldots, \quad \ldots \quad (25)$$

where A_n's are simple numerical constants and are given by

$$A_n = 2\pi i^n \int_0^{\pi}\int_0^{2\pi} [C_n(\theta, \phi)]^2 \frac{\sin \theta}{p_0^2} d\theta d\phi. \quad \ldots \quad (26)$$

Similarly, we obtain the expansion

$$e^{ik\rho(a \sin \theta \cos \phi \sin u \cos v + b \sin \theta \sin \phi \sin u \sin v + c \cos \theta \cos u)}$$

$$= A_0\psi_0(k\rho)+A_1\psi_1(k\rho)C_1(\theta, \phi; u, v)+\ldots$$
$$+A_n\psi_n(k\rho)C_n(\theta, \phi; u, v)+\ldots, \quad \ldots \quad (27)$$

the constants A_0, A_1, etc. having the same values as before.

The expression for V can therefore be written in the form

$$V = \sum_{n=0}^{\infty} A_n\psi_n(k\rho)\int_0^{\pi}\int_0^{2\pi} C_n(\theta, \phi; u, v)f(u, v)dudv. \quad \ldots \quad (28)$$

If now $C_n(\theta, \phi; u, v)$ be expanded in a series of the form (19) and if $f(u, v)$ be chosen to be

$$\sin u\, P_n^m(\cos u) \cos mv,$$

a solution of the wave equation in the co-ordinates ρ, θ, ϕ is at once obtained in the form

$$\psi_n(k\rho) C_n^m(\theta, \phi). \qquad \ldots \quad (29)$$

If on the other hand $f(u, v)$ is taken to be

$$\sin u\, P_n^m(\cos u) \sin mv,$$

another solution of the wave equation is obtained in the form

$$\psi_n(k\rho) S_n^m(\theta, \phi). \qquad \ldots \quad (30)$$

A number of interesting physical problems can be solved with the help of the solutions obtained above. For example, the non-stationary state of heat in an ellipsoid given by $\rho=1$, with the condition of zero temperature at the boundary, can be expressed in terms of (29) and (30), k being a root of the equation

$$\psi_n(k)=0.$$

Similarly, the periods of free oscillations of a gas, contained within the ellipsoidal shell $\rho=1$, are given by k which are the roots of the equation

$$\psi_n'(k)=0.$$

A memoir by the present writer on the many elegant and interesting properties of the functions introduced in this paper and their applications to physical problems involving ellipsoidal boundaries will be published shortly.

Some cases of Tidal oscillations in canals of variable section.

BY

Sasadhar Dasgupta.

1. Problems on seiches in lakes and tidal waves in estuaries have attracted considerable attention from mathematicians for a long time. Prof. Chrystal[1] and Lamb[2] have attempted to give a satisfactory mathematical theory of these phenomena. In view of the very interesting theoretical results obtained by these writers, I was led to study some more cases not considered by them.

Towards the end of the paper I have considered the second and higher order waves in a parabolic lake.

It may be remarked that the only case for which the second order tides have been determined is the one considered by Airy[3] and Mc. Cown[4] in which the section is uniformly rectangular throughout. As usual I find that the frequency of the "second order tide" is double that of the primary disturbance.

I am thankful to Dr. S. K. Banerji for the interest he has taken in the preparation of this paper.

Part I.

First Order Tides.

2. The free tidal oscillations in canals of variable section are determined by the equations

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{g}{b(x)} \cdot \frac{\partial}{\partial x} \left[\mathrm{S} . \frac{\partial \eta}{\partial x} \right], \qquad \dots (1)$$

$$\frac{\partial^2}{\partial t^2} [\mathrm{S}\xi] = g\mathrm{S} . \frac{\partial}{\partial x} \left[\frac{1}{b(x)} \cdot \frac{\partial}{\partial x} (\mathrm{S}\xi) \right], \qquad \dots (2)$$

$$\eta = -\frac{1}{b(x)} \frac{\partial}{\partial x} (\mathrm{S}\xi), \qquad \dots (3)$$

[1] Chrystal, "Some results in the mathematical theory of seiches" *Trans R. S. Edin.*, t. XLI, p. 599 (1905).

[2] Lamb's *Hydrodynamics*, 4th edition, p. 267.

[3] Airy, "Tides and Waves." *Ency. Metrop.*, Art. 192, (1845).

[4] Mc. Cown, "On the theory of long waves" *Phil. Mag.*, (5), t. XXXV, p. 250 (1893).

where S is the section of the canal at the point x, η the tidal elevation above the equilibrium level, $b(x)$ the breadth, and ξ the time integral of the displacement past the plane x up to the time t.

I now proceed to obtain the solution of these equations for various types of canals.

CASE I.

3. Suppose that the horizontal section of the canal is a parabola given by $b(x)=\frac{x^2}{2c}$ and that the depth is constant.

Assuming that $\eta \propto \cos(\sigma t+\epsilon)$ we see from (1) that

$$\frac{d^2\eta}{dx^2}+\frac{2}{x}\frac{d\eta}{dx}+k^2\eta=0 \qquad \ldots \quad (4)$$

where $k^2=\sigma^2/gh$.

The solution of (4) is given by

$$\eta=\text{A}x^{-\frac{1}{2}}\text{J}_{\frac{1}{2}}(kx)\cos(\sigma t+\epsilon)$$

$$=\text{A}x^{-\frac{1}{2}}\left(\frac{2}{\pi kx}\right)^{\frac{1}{2}}\sin(kx).\cos(\sigma t+\epsilon).$$

If the canal communicates with an open sea at its mouth $x=a$ at which tidal oscillations of the type $\eta=\text{C}\cos(\sigma t+\epsilon)$ are maintained, then

$$\eta=\text{C}\frac{a}{x}\cdot\frac{\sin(kx)}{\sin(ka)}\cos(\sigma t+\epsilon).$$

If the canal be closed at $x=a$, the admissible values of k are given by

$$\left|\frac{\partial\eta}{\partial x}\right|_{x=a}=0,$$

$$i.e., \ \frac{\partial}{\partial x}\left[x^{-\frac{1}{2}}\text{J}_{\frac{1}{2}}(kx)\right]_{x=a}=0,$$

or,

$$\tan(ka)=ka.$$

4. The depth being constant, let us inquire under what circumstances,

$$\frac{\partial^2\eta}{\partial t^2}=\frac{gh_0}{b(x)}\frac{\partial}{\partial x}\left(b(x)\frac{\partial\eta}{\partial x}\right) \quad \dots \quad (5)$$

has a solution of the form

$$\eta=\phi(x)\mathrm{F}\,[\psi(x)\pm\sigma t]. \quad \dots \quad (6)$$

where F is an arbitrary function and ϕ, ψ are definite functions of the argument x.

Now (5) can be written in the form

$$\frac{1}{gh}\frac{\partial^2\eta}{\partial t^2}=\frac{\partial^2\eta}{\partial x^2}+\frac{\partial\eta}{\partial x}\cdot\frac{\partial\log[b(x)]}{\partial x} \quad \dots \quad (7)$$

Substituting the value of η from (6) in (7) we have

$$\mathrm{F}''\left(\frac{\sigma^2}{gh}\phi-\phi\psi'^2\right)-\mathrm{F}'\left(2\phi'\psi'+\phi\psi''+\frac{\partial\log[b(x)]}{\partial x}\cdot\phi\psi'\right)$$

$$-\mathrm{F}\left(\phi''+\frac{\partial\log[b(x)]}{\partial x}\phi'\right)=0.$$

Since F is arbitrary :—

$$\psi'^2-\frac{\sigma^2}{gh}=0,$$

$$2\phi'\psi'+\phi\psi''+\frac{\partial\log[b(x)]}{\partial x}\cdot\phi\psi'=0,$$

$$\phi''+\frac{\partial\log b(x)}{\partial x}\phi'.=0.$$

$$\therefore\quad \psi(x)=\pm\frac{\sigma x}{(gh)^{\frac{1}{2}}}+\mathrm{C}_1,$$

$$\phi(x)=(\mathrm{B}x+\mathrm{C})^{-1},$$

$$b(x)=\mathrm{A}\,(\mathrm{B}x+\mathrm{C})^2.$$

If $\mathrm{C}_1=\mathrm{C}=0$ and also $\frac{\sigma^2}{gh}=k^2$, then assuming $\mathrm{F}=e^{i(kx\pm\sigma t)}$ we get the harmonic solution which we have already considered.

CASE II.

5. Assuming that both the bed and surface of the canal are sloping, so that $S=b_0h_0x^2$, the other circumstances being the same as in case I, we get from equation (1),

$$\frac{1}{x}\frac{\partial}{\partial x}\left[x^2\frac{\partial\eta}{\partial x}\right]+\frac{\sigma^2}{gh_0}\eta=0.$$

Putting $z^2=2x$ and $\frac{2\sigma^2}{gh_0}=k$, we have

$$\frac{d^2\eta}{dz^2}+\frac{3}{z}\frac{d\eta}{dz}+k\eta=0.$$

Therefore, $$\eta = Az^{-1}J_1(k^{\frac{1}{2}}z)\cos(\sigma t+\epsilon)$$

$$=A(2x)^{-\frac{1}{2}}J_1[(2kx)^{\frac{1}{2}}]\cos(\sigma t+\epsilon).$$

On determining the constant, we have

$$\eta = B\left(\frac{x}{a}\right)^{-\frac{1}{2}}\frac{J_1[(2kx)^{\frac{1}{2}}]}{J_1[(2ka)^{\frac{1}{2}}]}\cos(\sigma t+\epsilon).$$

Now if we assume that $\eta\propto\cos(n\sigma t)$ where n is arbitrary and $\sigma^2=\frac{gh_0}{2}$, a particular integral of

$$\frac{1}{x}\frac{\partial}{\partial x}\left[x^2\frac{\partial\eta}{\partial x}\right]+\frac{n^2}{2}\eta=0 \text{ is}$$

$$\eta=A\left[(2x)^{\frac{1}{2}}\right]^{-1}J_1\left[n(2x)^{\frac{1}{2}}\right]\cos n\sigma t$$

$$=-\frac{A}{n}\frac{\partial J_0[n(2x)^{\frac{1}{2}}]}{\partial x}.\cos n\sigma t.$$

$$=-\frac{A}{n\pi}\frac{\partial}{\partial x}\int_0^{\pi}\cos\left[n(2x)^{\frac{1}{2}}\cos\phi\right]\cos n\sigma t.\,d\phi$$

$$=-\frac{A}{2n\pi}\frac{\partial}{\partial x}\int_0^{\pi}\left[\cos n\{(2x)^{\frac{1}{2}}\cos\phi+\sigma t\}+\cos n\{(2x)^{\frac{1}{2}}\cos\phi-\sigma t\}\right]d\phi$$

Therefore we have as a solution :—

$$\eta=\frac{\partial}{\partial x}\int_0^\pi\left[F\{(2x)^{\frac{1}{2}}\cos\phi+\sigma t\}+F\{(2x)^{\frac{1}{2}}\cos\phi-\sigma t\}\right]d\phi$$

where F is a function capable of being expanded in a series of cosines.

To determine F we shall have to satisfy the initial condition. Suppose $\eta=\eta_0$ when $t=0$, then

$$\int\eta_0\,dx=2\int_0^\pi F[(2x)^{\frac{1}{2}}\cos\phi]d\phi.$$

If $\int\eta_0 dx=\eta_1$, then the determination of F involves the solution of the integral equation

$$\eta_1=2\int_0^\pi F\,[(2x)^{\frac{1}{2}}\cos\phi]d\phi.$$

CASE III.

6. If the surface is parabolic and the bed sloping, such that $h(x)=h_0.x$, $[b.(x)]^2=ax$, then the equation (1) becomes

$$\frac{1}{x^{\frac{1}{2}}}\frac{\partial}{\partial x}\left[x^{\frac{3}{2}}\frac{\partial\eta}{\partial x}\right]+\frac{\sigma^2}{gh_0}\eta=0.$$

Putting $z^2=2x$, and $k^2=\dfrac{2\sigma^2}{gh_0}$ we get

$$\frac{d^2\eta}{dz^2}+\frac{2}{z}\frac{d\eta}{dz}+k^2\eta=0.$$

Since η is to be finite at the origin

$$\eta=Az^{-\frac{1}{2}}J_{\frac{1}{2}}(kz)\cos(\sigma t+\epsilon)$$

$$=A\left[(2x)^{\frac{1}{2}}\right]^{-\frac{1}{2}}J_{\frac{1}{2}}[k(2x)^{\frac{1}{2}}]\cos(\sigma t+\epsilon).$$

Assuming that the bed is sloping, let us enquire under what circumstances (1) may have a solution of the form :

$$\eta=\phi(x)\ F\ (\psi(x)\pm\sigma t).$$

Following the same method as in case I, we get

$$\phi(x) = (Bx^{\frac{1}{2}}+C)^{-1}$$

and $h(x)=\dfrac{D[Bx^{\frac{1}{2}}+C]^2}{x}$.

CASE IV.

7. Let us now consider the case[1] when the breadth is constant, the longitudinal section through the x axis is parabolic being given by $h(x)=h_0\left(1-\dfrac{x^2}{a^2}\right)$ and the section perpendicular to the axis at any point a parabola of latus rectum, $4k$ say, where k varies from point to point and as can be easily seen is given by

$$k=\frac{b_0}{4h_0\left(1-\dfrac{x^2}{a^2}\right)}$$

When $x=\pm a$, the limiting parabolas at the extremities of the lake coincide with the bounding lines.

Putting $\qquad u=S.\xi,\ r=\displaystyle\int b(x)dx$ in (2) and (3) we have,

$$\frac{\partial^2 u}{\partial t^2}=g.\ S.\ h(x)\frac{\partial^2 u}{\partial r^2} \qquad \dots \quad (8)$$

$$\eta=-\frac{\partial u}{\partial r} \qquad \dots \quad (9)$$

where

$$S=\frac{4b_0h_0}{3}\left(1-\frac{x^2}{a^2}\right).$$

[1] Chrystal and Lamb have considered the section perpendicular to the x axis to be rectangular.

Substituting in (8) and assuming that $u \propto \cos(\sigma t+\epsilon)$ and putting $x=a\omega$, we have

$$(1-\omega^2)\frac{\partial^2 u}{\partial \omega^2}+cu=0, \qquad \text{where } c=\frac{3\sigma^2 a^2}{4gh_n} \qquad \ldots \quad (10)$$

This equation has the following solutions consistent with the boundary conditions :—

(a)
$$\xi=\frac{3A}{4b_0 h_n}\frac{C(c_{2s-1},\omega)}{(1-\omega^2)}\cos(\sigma_{2s-1}t+\epsilon).$$

$$\eta=-\frac{1}{b_n a}\frac{\partial u}{\partial \omega}=-\frac{A}{ab_n}\frac{\partial}{\partial \omega}\left[C(c_{2s-1},\omega)\right]\cos(\sigma_{2s-1}t+\epsilon).$$

where $C(c,\omega)$ is Chrystal's seiche cosine function and c_{2s-1} is a root of $C(c,1)=0$, it being$=2s.(2s-1)$ where s is an integer.

(b)
$$\xi=\frac{3B.S(c_{2s},\omega)}{4b_n h_n(1-\omega^2)}\cos(\sigma_{2s}t+\epsilon).$$

$$\eta=-\frac{B}{ab_n}\frac{\partial}{\partial \omega}\left[S(c_{2s},\omega)\right]\cos(\sigma_{2s}t+\epsilon)$$

where $S(c,\omega)$ is Chrystal's seiche sine function and c_{2s} is a root of $S(c,1)=0$, it being $=2s(2s+1)$ where s is an integer.

In either case the period of the n—nodal seiche is given by

$$T_n=\frac{2\pi}{\sigma_n}=\frac{\pi l}{\sqrt{n(n+1)gh_n}}\cdot\frac{\sqrt{3}}{2}, \quad \text{where } l=\text{length of the lake.}$$

This shows that the period of the n—nodal seiche is $\frac{\sqrt{3}}{2}$ times the corresponding period for a lake with tarnsverse rectangular section.

N.B.—If the longitudinal section be a convex parabola, the other circumstances being the same as in case IV we get the solution in terms of Chrystal's seiche hyperbolic sine and cosine functions. The period is however given by a similar expression.

CASE V.

8. If $b(x)=b_0$, $h(x)=h_0\sqrt{1-\frac{x^2}{a^2}}$, *i.e.*, if the longitudinal section is an ellipse whose major axis is in the equilibrium level, we have

$$\frac{\partial}{\partial x}\left[\sqrt{1-\frac{x^2}{a^2}}\frac{\partial\eta}{\partial x}\right]+\frac{\sigma^2}{gh_0}\eta=0 \qquad \dots \quad (11)$$

Putting $\frac{x}{a}=\sin 2z$, we get

$$\frac{d^2\eta}{dz^2}+2\theta_1\cos 2z.\eta=0, \text{ where } \theta_1=\frac{2\sigma^2a^2}{gh_0}.$$

To solve it assume $\eta=e^{\mu z}\sum_{n=-\infty}^{n=\infty}b_ne^{2niz}$

Substituting in (1) and equating the co-efficients of powers of e^{2iz} to zero, we obtain the system of equations

$$(\mu+2ni)^2b_n+\theta_1b_{n-1}+\theta_1b_{n+1}=0,$$

$$n=[\dots-2,-1,0,1,2\dots]$$

If we eliminate the co-efficients b_n etc. after dividing the typical equation by θ_1-4n^2 to secure convergence, we obtain the infinite determinant

$$\begin{vmatrix}
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \frac{-\theta_1}{4^2-\theta_1} & \frac{(i\mu+4)^2}{4^2-\theta_1} & \frac{-\theta_1}{4^2-\theta_1} & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & \frac{-\theta_1}{2^2-\theta_1} & \frac{(i\mu+2)^2}{2^2-\theta_1} & \frac{-\theta_1}{2^2-\theta_1} & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & \frac{-\theta_1}{0^2-\theta_1} & \frac{(i\mu)^2}{0^2-\theta_1} & \frac{-\theta_1}{0^2-\theta_1} & 0 & & \dots \\
\dots & 0 & 0 & 0 & \frac{-\theta_1}{2^2-\theta_1} & \frac{(i\mu-2)^2}{2^2-\theta_1} & \frac{-\theta_1}{2^2-\theta_1} & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & \frac{-\theta_1}{4^2-\theta_1} & \frac{(i\mu-4)^2}{4^2-\theta_1} & \frac{-\theta_1}{4^2-\theta_1} & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{vmatrix}=0$$

or

$$\Delta(i\mu)=0 \text{ say.}$$

Thus we get $\sin^2\left(\frac{1}{2}\pi i\mu\right)=\Delta(0)\sin^2\left(\frac{1}{2}\pi\sqrt{\theta_1}\right)$.

[See Whittaker's analysis p. 409].

This determines μ. Then b_n is given by $b_n=\frac{L_n}{L_0}b_0$ where L_n =co-factor of b_n in the determinant $\Delta(i\mu)$ and L_0=co-factor of b_0.

Thus if the canal communicates with an open sea in which tidal waves

$$\eta=C\cos(\sigma t+\epsilon)$$

are maintained, we have

$$\eta=\frac{Ce^{\mu z}\sum\limits_{n=-\infty}^{n=\infty}L_n e^{2niz}\cos(\sigma t+\epsilon)}{e^{\mu z_1}\sum\limits_{n=-\infty}^{n=\infty}L_n e^{2niz_1}}$$

where z_1 is given by $\sin 2z_1=\frac{c_1}{a}$, c_1 being the distance of the mouth of the canal from the origin.

Part II.

Second and Higher Order Tides.

9. Adopting the Lagrangian plan of making the co-ordinates refer to the individual particles of the fluid, the following equations can be easily established :—

$$\frac{\partial^2\xi}{\partial t^2}=g.\frac{1}{1+\frac{\partial\xi}{\partial x}}.\frac{\partial}{\partial x}\left[\frac{S}{b}\frac{\frac{\partial\xi}{\partial x}}{\left(1+\frac{\partial\xi}{\partial x}\right)}\right], \quad \dots (1)$$

$$\frac{\eta}{h}=-\frac{\frac{\partial\xi}{\partial x}}{1+\frac{\partial\xi}{\partial x}}, \quad \dots (2)$$

where the symbols used have the same meaning as before.

10. Let $b=b_0$ and $S=b_0h=b_0h_0\left(1-\frac{x^2}{a^2}\right)$, so that the longitudinal section is a parabola given by $h=h_0\left(1-\frac{x^2}{a^2}\right)$.

From (1), putting $\frac{x}{a}=z$ and neglecting the third order-terms we get

$$\frac{a^2}{gh_0}\frac{\partial^2\xi}{\partial t^2}=\frac{\partial}{\partial z}\left[(1-z^2)\frac{\partial\xi}{\partial z}\right]-\frac{2}{a}\frac{\partial\xi}{\partial z}\frac{\partial}{\partial z}\left[(1-z^2)\frac{\partial\xi}{\partial z}\right]$$

$$-\frac{1}{a}\frac{\partial\xi}{\partial z}\cdot\frac{\partial^2\xi}{\partial z^2}\qquad\ldots\quad(3)$$

Neglecting square terms in (3) we have

$$\frac{a^2}{gh_0}\frac{\partial^2\xi}{\partial t^2}=\frac{\partial}{\partial z}\cdot\left[(1-z^2)\frac{\partial\xi}{\partial z}\right].$$

Assuming that $\xi\propto e^{i\sigma_n t}$ and putting $\sigma_n^2=n(n+1)\frac{gh_0}{a^2}$ we have

$$\frac{\partial}{\partial z}\left[(1-z^2)\frac{\partial\xi}{\partial z}\right]+n(n+1)\xi=0.$$

This has the solution $\xi=CP_n(z)$, where n is determined by the condition that ξ is finite when $z=\pm1$ whence we see that n is integral.

For any integral value of n, σ_n is given by $\sigma_n^2=n(n+1)\frac{gh_0}{a^2}$.

Now reverting to equation (3), assume

$$\xi=CP_n(z)e^{i\sigma_n t}+\Sigma A_r P_r(z)e^{2i\sigma_n t},$$

where A_r is supposed small so that its square can be neglected.

Substituting in (3) and always neglecting A_r^2 etc., we have

$$-4n(n+1)\Sigma A_r P_r e^{2i\sigma_n t}=-\Sigma A_r r(r+1)P_r e^{2i\sigma_n t}$$

$$-\frac{2C^2}{a}\frac{\partial P_n}{\partial z}\frac{\partial}{\partial z}\left[(1-z^2)\frac{\partial P_n}{\partial z}\right]e^{2i\sigma_n t}$$

$$-\frac{C^2}{a}\frac{\partial P_n}{\partial z}(1-z^2)\frac{\partial^2 P_n}{\partial z^2}e^{2i\sigma_n t},$$

$$\text{or } \Sigma \left[4n(n+1)-r(r+1)\right] A_r P_r e^{2i\sigma_n t}$$

$$+\frac{2C^2}{a}\frac{\partial P_n}{\partial z}\cdot n(n+1)P_n e^{2i\sigma_n t}-\frac{C^2}{a}\frac{\partial P_n}{\partial z}\left[2z\frac{\partial P_n}{\partial z}\right.$$

$$\left.-n(n+1)P_n\right]e^{2i\sigma_n t}=0,$$

$$\text{or } \Sigma \left[4n(n+1)-r(r+1)\right] A_r P_r e^{2i\sigma_n t}$$

$$=\frac{C^2}{a}\left[2z\left(\frac{\partial P_n}{\partial z}\right)^2-3n(n+1)\frac{\partial P_n}{\partial z}\cdot P_n\right]e^{2i\sigma_n t}$$

we shall now expand $2z\left(\frac{\partial P_n}{\partial z}\right)^2-3n(n+1)P_n\frac{\partial P_n}{\partial z}$ in a series of zonal harmonics.

Let $f(z)=2z\left(\frac{\partial P_n}{\partial z}\right)^2-3n(n+1)P_n\frac{\partial P_n}{\partial z}$

$$=\frac{\partial P_n}{\partial z}\left[2P_1\{(2n-1)P_{n-1}+(2n-5)P_{n-3}+\ldots\}-3n(n+1)P_n\right]$$

Now
$$P_1P_{n-1}=\frac{n}{2n-1}P_n+\frac{n-1}{2n-1}P_{n-2}$$

$$P_1P_{n-3}=\frac{n-2}{2n-5}P_{n-2}+\frac{n-3}{2n-5}P_{n-4}$$

$$P_1P_{n-5}=\frac{n-4}{2n-9}P_{n-4}+\frac{n-5}{2n-9}P_{n-6}$$

etc.

$$\therefore f(z)=\frac{\partial P_n}{\partial z}\left[\{2n-3n(n+1)\}P_n+2(n-3)P_{n-2}+2(2n-7)P_{n-4}+\ldots\right]$$

$$=[B_{n-1}P_{n-1}+B_{n-3}P_{n-3}+\ldots][D_nP_n+D_{n-2}P_{n-2}+\ldots]$$

$$=C_{2n-1}P_{2n-1}+C_{2n-3}P_{2n-3}+\ldots+C_1P_1$$

(for all values of n even or odd, the last terms of $f(z)$ will contain P_1 as can be easily seen). Here

$$D_n=-3n^2-n,$$

$$D_{n-2}=2(2n-3),$$

$$D_{n-4}=2(2n-7),$$

etc., and

$$B_{n-1}=2n-1,$$

$$B_{n-3}=2n-5,$$

$$B_{n-5}=2n-9,$$

etc.

Now using Adam's result for the expansion of the product of any two Legendre's co-efficients in terms of the Lengendre's co-efficients, we have

$$C_{2n-1}=D_nB_{n-1}\,\frac{1\cdot3\cdot5\ldots(2n-3)}{(n-1)\,!}\cdot\frac{(n+1)(n+2)\ldots(2n-1)}{(2n+1)(2n+3)\ldots(4n-1)}\times(4n-1),$$

$$C_{2n-3}=D_nB_{n-1}\,\frac{1.3.5\ldots(2n-5)}{(n-2)\,!}\cdot\frac{n(n+1)\ldots(2n-2)}{(2n-1)(2n+1)\ldots(4n-3)}\times(4n-5)$$

$$+D_nB_{n-3}\,\frac{1.3.5\ldots(2n-5)}{(n-3)\,!}\cdot\frac{(n+1)(n+2)\ldots(2n-3)}{(2n+1)(2n+3)\ldots(4n-5)}\times(4n-5)$$

$$+B_{n-1}D_{n-2}\,\frac{1\cdot3\ldots(2n-5)}{(n-2)\,!}\cdot\frac{n(n+1)\ldots(2n-3)}{(2n-1)(2n+1)\ldots(4n-5)}\times(4n-5),$$

$C_{2n-5}=$etc.

Thus
$$A_1=\frac{c^2}{a}\,\frac{C_1}{[4n(n+1)-1\cdot2]}\,.$$

$$A_3=\frac{c^2}{a}\,\frac{C_3}{[4n(n+1)-3.4]}\,,$$

$$A_{2n-(2j-1)}=\frac{c^2}{a}\,\frac{C_{2n-(2j-1)}}{[4n(n+1)-(2n-2j+2)(2n-2j+1)]}\,.$$

The nature of the solution obtained above shows that if ξ is calculated to 2nd order of approximation we get tides of the 2nd order being proportional to c^2 and the frequency being double that of the primary disturbance. Continuing the approximation we obtain tides of higher orders of frequencies 3, 4...times that of the primary.

11. The following particular cases are interesting:—

1. When $n=1$,

$$\xi=CP_1(z)e^{i\sigma_1 t}-\frac{2}{3}\,\frac{c^2}{a}\cdot P_1(z)e^{2i\sigma_1 t}$$

and

σ_1 is given by $\sigma_1^2=1\cdot2\cdot\dfrac{gh_0}{a^2}$.

2. When $n=2$,

$$\xi=CP_2(z)e^{i\sigma_2 t}-\frac{c^2}{a}\left[\frac{27}{55}\cdot P_1(z)+\frac{21}{10}\cdot P_3(z)\right]e^{2i\sigma_2 t}\,.$$

and

σ_2 is given by $\sigma_2^2=2.3.\dfrac{gh_0}{a^2}$.

3. When $n=3$,

$$\xi=CP_3(z)e^{i\sigma_3 t}-\frac{c^2}{a}\left[\frac{82}{161}\,P_1(z)+\frac{197}{126}\,P_3(z)+\frac{250}{63}\,P_5(z)\right]\times e^{2i\sigma_3 t}$$

and σ_3 is given by $\sigma_3^2=3.4.\dfrac{gh_0}{a^2}$.

The Stress-Equations of Equilibrium

BY

Satyendranath Basu.

It was shown by Mitchell that the six stress co-efficients in an isotropic medium satisfy six equations of the type :—

$$\nabla^2 X_x + \frac{1}{1+\sigma} \frac{\partial^2 \Theta}{\partial x^2} = 0, \qquad \nabla^2 Y_z + \frac{1}{1+\sigma} \frac{\partial^2 \Theta}{\partial y \partial z} = 0.$$

... ...

... ...

These equations however have not been used for solving the general problems of Elasticity. It is shown here, that the equations can be successfully integrated, in the case of a semi-infinite body bounded by a plane. In the case of the sphere the equations can be conveniently transformed, in a different form, which then admit of integration in an infinite series of spherical harmonics.

(1) *The semi-infinite solid bounded by* $z=0$.

The surface tractions X_z, Y_z, Z_z, are supposed to have given values over the plane $z=0$.

Consider the equations

$$\nabla^2 X_z + \frac{1}{1+\sigma} \frac{\partial^2 \Theta}{\partial x \partial z} = 0, \qquad \nabla^2 Z_z + \frac{1}{1+\sigma} \frac{\partial^2 \Theta}{\partial z^2} = 0,$$

$$\nabla^2 Y_z + \frac{1}{1+\sigma} \frac{\partial^2 \Theta}{\partial y \partial z} = 0.$$

Since $\odot$ is a harmonic function the general solution can be written as

$$\left.\begin{aligned} X_z &= -\frac{1}{2(1+\sigma)}\, z\, \frac{\partial \odot}{\partial x} + X_{z_0}, \\ Y_z &= -\frac{1}{2(1+\sigma)}\, z\, \frac{\partial \odot}{\partial y} + Y_{z_0}, \\ Z_z &= -\frac{1}{2(1+\sigma)}\, z\, \frac{\partial \odot}{\partial z} + Z_{z_0}. \end{aligned}\right\} \quad \ldots \quad (1)$$

where X_{z_0}, Y_{z_0}, Z_{z_0} are harmonic functions which have given values X_z, Y_z, Z_z, over the plane $z=0$.

The functions are therefore uniquely determined; they are in fact

$$X_{z_0} = \frac{1}{2\pi}\frac{\partial}{\partial z}\iint \frac{X_z}{r}\, dx\,dy, \quad Y_{z_0} = \frac{1}{2\pi}\frac{\partial}{\partial z}\iint \frac{Y_z}{r}\, dx\,dy,$$

$$Z_{z_0} = \frac{1}{2\pi}\frac{\partial}{\partial z}\iint \frac{Z_z}{r}\, dx\,dy.$$

also since

$$\frac{\partial X_z}{\partial x} + \frac{\partial Y_z}{\partial y} + \frac{\partial Z_z}{\partial z} = 0,$$

we have from (1)

$$-\frac{1}{2(1+\sigma)}\frac{\partial \odot}{\partial z} + \frac{1}{2\pi}\frac{\partial}{\partial z}\left(\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}\right) = 0.$$

where

$$L = \iint \frac{X_z}{r}\, dx\,dy, \quad M = \iint \frac{Y_z}{r}\, dx\,dy, \quad N = \iint \frac{Z_z}{r}\, dx\,dy.$$

and

$$\odot = \frac{1+\sigma}{\pi}\left[\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}\right].$$

Thus X_z, Y_z, Z_z, $\odot$, are all determined.

The solution may be afterwards completed, and U, V, W found out as in Cerrutti's method.

(2) *The problem of the sphere.*

Consider the three equations

$$\nabla^2 X_x + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial x^2} = 0,$$

$$\nabla^2 X_y + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial x \partial y} = 0,$$

$$\nabla^2 X_z + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial x \partial z} = 0.$$

Multiplying by x, y, z, and adding we have

$$[x\nabla^2 X_x + y\nabla^2 X_y + z\nabla^2 X_z$$

$$+ \frac{1}{1+\sigma}\left[x\frac{\partial^2 \Theta}{\partial x^2} + y\frac{\partial^2 \Theta}{\partial x \partial y} + z\frac{\partial^2 \Theta}{\partial x \partial z}\right] = 0,$$

or,

$$\nabla^2 (xX_x + yX_y + zX_z) + \frac{1}{1+\sigma}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\frac{\partial \Theta}{\partial x}$$

$$-2\left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\right) = 0.$$

Now since

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = 0,$$

we have

$$\left.\begin{aligned} &\nabla^2 (rX_r) + \frac{1}{1+\sigma}\frac{\partial}{\partial x}\left(r\frac{\partial \Theta}{\partial r} - \Theta\right) = 0. \\ &\text{Similarly we have} \\ &\nabla^2 (rY_r) + \frac{1}{1+\sigma}\frac{\partial}{\partial y}\left(r\frac{\partial \Theta}{\partial r} - \Theta\right) = 0, \\ &\nabla^2 (rZ_r) + \frac{1}{1+\sigma}\frac{\partial}{\partial z}\left(r\frac{\partial \Theta}{\partial r} - \Theta\right) = 0, \end{aligned}\right\} \quad \cdots \quad \cdots \quad (2)$$

where $r\frac{\partial \Theta}{\partial r} - \Theta$ is also a harmonic function.

The form of the equations is exactly similar to the preceding equations.

(3) The sphere of radius a, has given tractions X_r, Y_r, Z_r over the surface.

Assuming $\odot = \Sigma \odot_n$,

where $\odot_n$ is a solid homogeneous harmonic of the nth degree,

$$r\frac{\partial \odot}{\partial r} - \odot = \Sigma(n-1)\odot_n.$$

also remembering that

$\nabla^2(r^2-a^2)F_n = 2(2n+3)F_n$, where F_n is a solid homogeneous harmonic of the nth degree, we see that the solutions of the equation (2) can be expressed in the form

$$rX_r = -\frac{1}{2(1+\sigma)}\Sigma\frac{(n-1)}{2n+1}(r^2-a^2)\frac{\partial \odot_n}{\partial x} + aX_{r,n},$$

$$rY_r = -\frac{1}{2(1+\sigma)}\Sigma\frac{n-1}{2n+1}(r^2-a^2)\frac{\partial \odot_n}{\partial y} + aY_{r,n},$$

$$rZ_r = -\frac{1}{2(1+\sigma)}\Sigma\frac{n-1}{2n+1}(r^2-a^2)\frac{\partial \odot_n}{\partial z} + aZ_{r,n}.$$

where $X_{r,n}$, $Y_{r,n}$, $Z_{r,n}$, are harmonic functions which have given values over the surface of the sphere $r=a$, and hence are completely determined. If

$$X_{r,n} = \Sigma X_n,\ Y_{r,n} = \Sigma Y_n.\ Z_{r,n} = \Sigma Z_n\ ;$$

we have

$$rX_r = -\frac{(r^2-a^2)}{2(1+\sigma)}\Sigma\frac{n-1}{2n+1}\frac{\partial \odot_n}{\partial x} + a\Sigma X_n$$

$$rY_r = -\frac{(r^2-a^2)}{2(1+\sigma)}\Sigma\frac{n-1}{2n+1}\frac{\partial \odot_n}{\partial y} + a\Sigma Y_n$$

$$rZ_r = -\frac{(r^2-a^2)}{2(1+\sigma)}\Sigma\frac{n-1}{2n+1}\frac{\partial \odot_n}{\partial z} + a\Sigma Z_n.$$

Again

$$\frac{\partial}{\partial x}(rX_r)+\frac{\partial}{\partial y}(rY_r)+\frac{\partial}{\partial z}(rZ_r)$$

$$=x\left[\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}\right]+y\left[\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}\right]$$

$$+z\left[\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}\right]+X_x+Y_y+Z_z=\odot$$

it follows that

$$\odot=-\frac{1}{(1+\sigma)}\Sigma\frac{n-1}{(2n+1)}\left[x\frac{\partial\odot_n}{\partial x}+y\frac{\partial\odot_n}{\partial y}+z\frac{\partial\odot_n}{\partial z}\right]$$

$$+a\Sigma\left(\frac{\partial X_n}{\partial x}+\frac{\partial Y_n}{\partial x}+\frac{\partial Z_n}{\partial z}\right)$$

or

$$\odot=-\frac{1}{1+\sigma}\Sigma\frac{n(n-1)}{2n+1}\odot_n+a\Sigma\left(\frac{\partial X_n}{\partial x}+\frac{\partial Y_n}{\partial y}+\frac{\partial Z_n}{\partial z}\right).$$

So that

$$\Sigma\odot_n\left[\frac{(1+\sigma)(2n+1)+n(n-1)}{(2n+1)(1+\sigma)}\right]=a\Sigma\Psi_n.$$

where

$$\Psi_n=\frac{\partial X_{n+1}}{\partial x}+\frac{\partial Y_{n+1}}{\partial y}+\frac{\partial Z_{n+1}}{\partial z}$$

So that

$$\odot_n=\frac{a(2n+1)(1+\sigma)}{(1+\sigma)(2n+1)+n(n-1)}.\Psi_n.$$

Thus X_r, Y_r, Z_r, $\odot$, are all determined in terms of the known value of X_r, Y_r, Z_r on the surface.

On a special square matrix of order six

BY

C. E. CULLIS

Summary.—The matrix ψ defined in equation (2) is one which occurs very frequently. Its rank and its reciprocal are determined in Art. 1. The remaining articles contain illustrations of its use. In Arts 4 and 5 the stress-strain relations of an isotropic body are determined without assuming the existence of a strain-energy function, the general argument being a simplification of that contained in Chapter VI of *Love's Elasticity*.

1. *Homogeneous linear transformations of the variables in the matrix*

$$[x^2, y^2, z^2, yz, zx, xy].$$

If we use the notations

$$\phi=[abc]_{123}, \quad \Delta=\det\phi=(abc)_{123}, \qquad \dots \quad (1)$$

$$\psi=[e]_6^6=\begin{bmatrix} a_1^2, & b_1^2, & c_1^2, & 2b_1c_1, & 2c_1a_1, & 2a_1b_1 \\ a_2^2, & b_2^2, & c_2^2, & 2b_2c_2, & 2c_2a_2, & 2a_2b_2 \\ a_3^2, & b_3^2, & c_3^2, & 2b_3c_3, & 2c_3a_3, & 2a_3b_3 \\ a_2a_3, & b_2b_3, & c_2c_3, & b_2c_3+b_3c_2, & c_2a_3+c_3a_2, & a_2b_3+a_3b_2 \\ a_3a_1, & b_3b_1, & c_3c_1, & b_3c_1+b_1c_3, & c_3a_1+c_1a_3, & a_3b_1+a_1b_3 \\ a_1a_2, & b_1b_2, & c_1c_2, & b_1c_2+b_2c_1, & c_1a_2+c_2a_1, & a_1b_2+a_2b_1 \end{bmatrix}. \qquad \dots \quad (2)$$

and apply to the variables x, y, z the transformation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1, b_1, c_1 \\ a_2, b_2, c_2 \\ a_3, b_3, c_3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \qquad \dots \quad (A)$$

we obtain

$$[x^2, y^2, z^2, yz, zx, xy]=[X^2, Y^2, Z^2, YZ, ZX, XY]\,\overline{e}_6^6. \qquad \dots \quad (3)$$

When $c_1=\lambda a_1+\mu b_1$, $c_2=\lambda a_2+\mu b_2$, $c_3=\lambda a_3+\mu b_3$, the minor of ψ formed with its 1st, 5th and 6th vertical rows becomes degenerate, and consequently all simple minor determinants of that minor matrix are divisible by Δ. In fact the affected minor determinants of ψ of order 3 belonging to that minor, when arranged according to the scheme

$$(123, 156, 146, 145, 256, 246, 245, 356, 346, 345,$$

$$456, 234, 235, 236, 134, 135, 136, 124, 125, 126),$$

form the matrix

$$\Delta.[-4a_1a_2a_3,\ a_1^3, -a_1^2a_2, -a_1^2a_3,\ a_1a_2^2, -a_2^3,\ a_2^2a_3, -a_1a_3^2, -a_2a_2^3,\ a_3^3\ a_1a_2a_3,\ 0, -2a_2a_3^2,\ 2a_2^2a_3,\ 2a_1a_3^2, 0,\ 2a_1^2a_3, 2a_1a_2^2, -2a_1^2a_2, 0], \quad \ldots \quad (4)$$

and the co-factors in ψ of these respective determinants form the matrix

$$\Delta.\ [-A_1A_2A_3,\ A_1^3, -A_1^2A_2, -A_1^2A_3,\ A_1A_2^2, -A_2^3,\ A_2^2A_3, -A_1A_3^2, -A_2A_3^2,\ A_3^3,\ 2A_1A_2A_3,\ 0, -A_2A_3^2,\ A_2^2A_3,\ A_1A_3^2,\ 0, A_1^2A_3, A_1A_2^2, -A_1^2A_2, 0]. \quad \ldots \quad (4')$$

where $[ABC]_{123}$ is the reciprocal of $[abc]_{123}$. Hence by expanding the determinant $[c]_6^6$ in terms of the minor determinants of order 3 belonging to the 1st, 5th and 6th vertical rows, we see that

$$[c]_6^6=\det\psi=\Delta^4. \quad \ldots \quad (5)$$

It follows that ψ is undegenerate or degenerate according as ϕ is undegenerate or degenerate.

It is easily seen in a similar way that when ϕ is degenerate, every vertical minor of ψ formed with four vertical rows is degenerate, and therefore the rank of ψ cannot exceed 3. It then follows from (4′) that when ϕ has rank 2, the matrix ψ has rank 3. It is moreover obvious that when ϕ has rank 1 or 0, the matrix ψ has the same rank. Therefore the rank of ψ depends on the rank of ϕ in the way shown in the following scheme.

Rank of ϕ	3	2	1	0
Rank of ψ	6	3	1	0

Again if $[ABC]_{123}$ is the reciprocal of $[abc]_{123}$, and if we write

$$\Psi=[E]_6^6$$

$$=\begin{bmatrix} A_1^2, & B_1^2, & C_1^2, & B_1C_1, & C_1A_1, & A_1B_1 \\ A_2^2, & B_2^2, & C_2^2, & B_2C_2, & C_2A_2, & A_2B_2 \\ A_3^2, & B_3^2, & C_3^2, & B_3C_3, & C_3A_3, & A_3B_3 \\ 2A_2A_3, & 2B_2B_3, & 2C_2C_3, & B_2C_3+B_3C_2, & C_2A_3+C_3A_2, & A_2B_3+A_3B_2 \\ 2A_3A_1, & 2B_3B_1, & 2C_3C_1, & B_3C_1+B_1C_3, & C_3A_1+C_1A_3, & A_3B_1+A_1B_3 \\ 2A_1A_2, & 2B_1B_2, & 2C_1C_2. & B_1C_2+B_2C_1. & C_1A_2+C_2A_1. & A_1B_2+A_2B_1 \end{bmatrix}, \tag{2'}$$

we see by direct multiplication that

$$[e]_6^6\ \underline{E}_6^6=\underline{E}_6^6\ [e]_6^6=\Delta^2[1]_6^6\ ; \qquad \ldots (6)$$

and this shows that

the reciprocal of $[e]_6^6$ *is* $\Delta^2\ [E]_6^6$.

When $\Delta\neq 0$, we deduce from (A) the inverse transformation

$$\Delta\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} A_1, & A_2, & A_3 \\ B_1, & B_2, & B_3 \\ C_1, & C_2, & C_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \ldots (A')$$

from which it follows that

$$\Delta^2.\ [X^2, Y^2, Z^2, YZ, ZX, XY]=[x^2, y^2, z^2, yz, zx, xy].\ [E]_6^6, \tag{3'}$$

as could have been deduced from (3) by prefixing $[E]_6^6$ on both sides.

Note.—When ϕ is undegenerate and

$$\begin{bmatrix} A_1, & A_2, & A_3 \\ B_1, & B_2, & B_3 \\ C_1, & C_2, & C_3 \end{bmatrix} \text{ is the } \textit{inverse} \text{ of } \begin{bmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{bmatrix},$$

the matrix Ψ defined by (2′) is the *inverse* of the matrix ψ defined by (2), and the two mutually inverse transformations

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \quad \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} A_1, & A_2, & A_3 \\ B_1, & B_2, & B_3 \\ C_1, & C_2, & C_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

lead to the two mutually inverse transformations

$$[x^2, y^2, z^2, yz, zx, xy] = [X^2, Y^2, Z^2, YZ, ZX, XY].\ [e]_6^6.$$

$$[X^2, Y^2, Z^2, YZ, ZX, XY,] = [x^2, y^2, z^2, yz, zx, xy].\ [E]_6^6.$$

2. *Rank of the matrix*

$$M_5 = f(x, y, z) = \begin{bmatrix} x_1^2, & y_1^2, & z_1^2, & y_1z_1, & z_1x_1, & x_1y_1 \\ x_2^2, & y_2^2, & z_2^2, & y_2z_2, & z_2x_2, & x_2y_2 \\ x_3^2, & y_3^2, & z_3^2, & y_3z_3, & z_3x_3, & x_3y_3 \\ x_4^2, & y_4^2, & z_4^2, & y_4z_4, & z_4x_4, & x_4y_4 \\ x_5^2, & y_5^2, & z_5^2, & y_5z_5, & z_5x_5, & x_5y_5 \end{bmatrix},$$

where $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots (x_5, y_5, z_5)$ *are the (projective) co-ordinates of five distinct points* $P_1, P_2, \ldots P_5$ *in homogeneous 2-way space.*

If a point P has co-ordinates (x, y, z) with respect to one triangle of reference, and co-ordinates (X, Y, Z) with respect to any other triangle of reference, there exists a transformation of the form (A) in which $\Delta \neq 0$, and we have

$$f(x, y, z) = f(X, Y, Z)\ [e]_6^6,$$

where $[e]_6^6$ is undegenerate. We conclude that the rank of M_5 is the same whatever the triangle of reference may be.

If all the five points $P_1, P_2, \ldots P_5$ are collinear, we can take the straight line on which they lie to be the side $x=0$ of the triangle of reference, and P_1 and P_2 to be corners of the triangle of reference, and suppose the co-ordinates of $P_1, P_2, \ldots P_5$ to be $(0, b, 0)$,

$(0, 0, c)$, $(0, \beta_1, \gamma_1)$, $(0, \beta_2, \gamma_2)$, $(0, \beta_3, \gamma_3)$, when $b, c, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3$, are all different from 0. Then M_5 has same rank as

$$\begin{matrix} b^2, & 0, & \beta_1^2, & \beta_2^2, & \beta_3^2 \\ 0, & c^2, & \gamma_1^2, & \gamma_2^2, & \gamma_3^2 \\ 0, & 0, & \beta_1\gamma_1, & \beta_2\gamma_2, & \beta_3\gamma_3 \end{matrix}$$

Thus in this case M_5 has rank 3.

If four but not five of the points $P_1, P_2, \ldots P_5$ are collinear, we can choose the triangle of reference so that their co-ordinates are $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, $(0, \beta_1, \gamma_1)$, $(0, \beta_2, \gamma_2)$, where each letter denotes a non-zero quantity. Then M_5 has the same rank as

$$\begin{matrix} a^2, & 0, & 0, & 0, & 0 \\ 0, & b^2, & 0, & \beta_1^2, & \beta_2^2 \\ 0, & 0, & c, & \gamma_1^2, & \gamma_2^2 \\ 0, & 0, & 0, & \beta_1\gamma_1, & \beta_2\gamma_2 \end{matrix}$$

Thus in this case M_5 has rank 4.

If three but not four of the points $P_1, P_2, \ldots P_5$ are collinear, we can choose the triangle of reference so that their co-ordinates are $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, $(0, \beta, \gamma)$, (x, y, z), where $a, b, c, \beta, \gamma, x$ are non-zero quantities, and y and z are not both zero. Then M_5 has the same rank as the matrix

$$\begin{matrix} a^2, & 0, & 0, & 0, & 0, & 0 \\ 0, & b^2, & 0, & 0, & 0, & 0 \\ 0, & 0, & c^2, & 0, & 0, & 0 \\ 0, & \beta^2, & \gamma^2, & \beta\gamma, & 0, & 0 \\ x^2, & y^2, & z^2, & yz, & zx, & xy \end{matrix}$$

which is 4 plus the rank of $[z, y]$. Thus in this case M_5 has rank 5.

If no three of the points $P_1, P_2, \ldots P_5$ are collinear, we can choose the triangle of reference so that their co-ordinates are $(a, 0, 0)$,

$(0, b, 0)$, $(0, 0, c)$, $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, where each letter denotes a non-zero quantity. Then M_5 has the same rank as the matrix

$$\begin{bmatrix} a^2, & 0, & 0, & 0, & 0, & 0 \\ 0, & b^2, & 0, & 0, & 0, & 0 \\ 0, & 0, & c^2, & 0, & 0, & 0 \\ \alpha_1^2, & \beta_1^2, & \gamma_1^2, & \beta_1\gamma_1, & \gamma_1\alpha_1, & \alpha_1\beta_1 \\ \alpha_2^2, & \beta_2^2, & \gamma_2^2, & \beta_2\gamma_2, & \gamma_2\alpha_2, & \alpha_2\beta_2 \end{bmatrix}.$$

Since the simple minor determinants of the matrix $\begin{vmatrix} \beta_1\gamma_1, & \gamma_1\alpha_1, & \alpha_1\beta_1 \\ \beta_2\gamma_2, & \gamma_2\alpha_2, & \alpha_2\beta_2 \end{vmatrix}$

cannot all vanish, we see that in this case M_5 has rank 5.

Thus in all cases the rank of the matrix $f(x, y, z)$ is given by the following scheme

Maximum number of collinear points	2	3	4	5
Rank of M_5	5	5	4	3

NOTE 1.—For the matrix $\mathsf{M}_4 = \begin{bmatrix} x_1^2, & y_1^2, & z_1^2, & y_1z_1, & z_1x_1, & x_1y_1 \\ x_2^2, & y_2^2, & z_2^2, & y_2z_2, & z_2x_2, & x_2y_2 \\ x_3^2, & y_3^2, & z_3^2, & y_3z_3, & z_3x_3, & x_3y_3 \\ x_4^2, & y_4^2, & z_4^2, & y_4z_4, & z_4x_4, & x_4y_4 \end{bmatrix}$

we have the following scheme.

Maximum number of collinear points	2	3	4
Rank of M_4	4	4	3

Note 2.—The rank of the matrix

$$M_3=\begin{bmatrix} x_1^2, & y_1^2, & z_1^2, & y_1z_1, & z_1x_1, & x_1y_1 \\ x_2^2, & y_2^2, & z_2^2, & y_2z_2, & z_2x_2, & x_2y_2 \\ x_3^2, & y_3^2, & z_3^2, & y_3z_3, & z_3x_3, & x_3y_3 \end{bmatrix}$$

is always 3, when the three points P_1, P_2, P_3 are distinct.

3. *Quadric curve or curves through five given points in homogeneous 2-way space.*

If $P_1, P_2, \ldots P_5$ are five distinct given points whose (projective) co-ordinates are (x_1, y_1, z_1), $(x_2, y_2, z_2), \ldots (x_5, y_5, z_5)$, then

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

will be a quadric curve passing through them if and only if the matrix of the co-efficients satisfies the equation

$$M\cdot\begin{bmatrix} a \\ b \\ c \\ f \\ g \\ h \end{bmatrix}=0, \text{ where } M=\begin{bmatrix} x_1^2, & y_1^2, & z_1^2, & 2y_1z_1, & 2z_1x_1, & 2x_1y_1 \\ x_2^2, & y_2^2, & z_2^2, & 2y_2z_2, & 2z_2x_2, & 2x_2y_2 \\ x_3^2, & y_3^2, & z_3^2, & 2y_3z_3, & 2z_3x_3, & 2x_3y_3 \\ x_4^2, & y_4^2, & z_4^2, & 2y_4z_4, & 2z_4x_4, & 2x_4y_4 \\ x_5^2, & y_5^2, & z_5^2, & 2y_5z_5, & 2z_5x_5, & 2x_5y_5 \end{bmatrix}.$$

By article 2 the possible ranks of M are 5, 4 and 3; and therefore the number of unconnected non-zero solutions of the equation is either 1 or 2 or 3. If we disregard the quadric curve of rank 0, we see that:

(1) There is always at least one quadric curve passing though the 5 points.

(2) Except when 4 of the points are collinear, there is only one quadric curve passing through the 5 points.

(3) When 4 of the points are collinear, but not all 5 of them, there are exactly two unconnected quadric curves passing through the 5 points.

(4) When all 5 of the points are collinear, there are three and only three unconnected quadric curves passing through the 5 points.

In case (3) a quadric curve through the 5 points consists of the straight line on which 4 of the points lie and a straight line passing through the other point. In case (4) a quadric curve through the 5 points consists of the straight line on which the 5 points lie and one other straight line.

Note.—*Conic or conics through five given coplanar points in common 3-way space.*

The foregoing results are of course true when (x_1, y_1, z_1), (x_2, y_2, z_2), (x_5, y_5, z_5) are the projective co-ordinates of five distinct given coplanar points with respect to a reference framework lying in their plane, the quadric curves being now conics. The points at infinity may as usual be taken to be those for which $z=0$; and some or all the given points may lie on the line at infinity. The 5 given points determine a conic uniquely except when 4 of them are collinear.

4. *The stress-strain relations for an isotropic solid.*

Consider any body, such as a solid, which is slightly strained from a state of zero stress. With the usual notations for the components of strain and stress with reference to rectangular axes (OX, OY, OZ) let

$$[e_1, e_2, e_3, e_4, e_5, e_6]=[e_{xx}, e_{yy}, e_{zz}, \tfrac{1}{2}e_{yz}, \tfrac{1}{2}e_{zx}, \tfrac{1}{2}e_{xy}],$$

$$[E_1, E_2, E_3, E_4, E_5, E_6]=[X_x, Y_y, Z_z, Y_z, Z_x, X_y];$$

and let the stress-strain relations for those axes of co-ordinates be

$$\overline{\underline{E}}_6 = |c|_6^6 \, \overline{\underline{e}}_6. \qquad \ldots (7)$$

Let (OX′, OY′, OZ′) be any other set of rectangular axes through O, the direction-cosines of OX′, OY′, OZ′ with reference to the axes (OX, OY, OZ) being respectively (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3). When this second set of co-ordinate axes is used, let $e_1, e_2, \ldots e_6$, E_1, $E_2, \ldots E_6$ be replaced by e_1', e_2', $\ldots e_6'$, E_1', $E_2', \ldots E_6'$, and let the stress-strain relations be

$$\overline{\underline{E'}}_6 = [c']_6^6 \, \overline{\underline{e'}}_6 \qquad \ldots (7')$$

Then the body is isotropic (in its unstrained state) if and only if

$$[c']_6^6 = [c]_6^6$$

for all choices of the second set of axes.

Now if

$$
[\omega]^6_6 = \begin{bmatrix}
l_1^2, & m_1^2, & n_1^2, & 2m_1n_1, & 2n_1l_1, & 2l_1m_1 \\
l_2^2, & m_2^2, & n_2^2, & 2m_2n_2, & 2n_2l_2, & 2l_2m_2 \\
l_3^2, & m_3^2, & n_3^2, & 2m_3n_3, & 2n_3l_3, & 2l_3m_3 \\
l_2l_3, & m_2m_3, & n_2n_3, & m_2n_3+m_3n_2, & n_2l_3+n_3l_2, & l_2m_3+l_3m_2 \\
l_3l_1, & m_3m_1, & n_3n_1, & m_3n_1+m_1n_3, & n_3l_1+n_1l_3, & l_3m_1+l_1m_3 \\
l_1l_2, & m_1m_2, & n_1n_2, & m_1n_2+m_2n_1, & n_1l_2+n_2l_1, & l_1m_2+l_2m_1
\end{bmatrix}. \qquad \dots (8)
$$

$$
\overline{\underline{\Omega}}{}^6_6 = \begin{bmatrix}
l_1^2, & l_2^2, & l_3^2, & 2l_2l_3, & 2l_3l_1, & 2l_1l_2 \\
m_1^2, & m_2^2, & m_3^2, & 2m_2m_3, & 2m_3m_1, & 2m_1m_2 \\
n_1^2, & n_2^2, & n_3^2, & 2n_2n_3, & 2n_3n_1, & 2n_1n_2 \\
m_1n_1, & m_2n_2, & m_3n_3, & m_2n_3+m_3n_2, & m_3n_1+m_1n_3, & m_1n_2+m_2n_1 \\
n_1l_1, & n_2l_2, & n_3l_3, & m_3n_1+m_1n_3, & n_3l_1+n_1l_3, & n_1l_2+n_2l_1 \\
l_1m_1, & l_2m_2, & l_3m_3, & m_1n_2+m_2n_1, & l_3m_1+l_1m_3, & l_1m_2+l_2m_1
\end{bmatrix}. \qquad \dots (8')
$$

we have

$$[\omega]^6_6 \, \overline{\underline{\Omega}}{}^6_6 = \overline{\underline{\Omega}}{}^6_6 \, [\omega]^6_6 = [1]^6_6 \,.$$

i.e. $[\omega]^6_6$ and $[\Omega]^6_6$ are two mutually inverse undegenerate square matrices; and the formulae for the transformation of strain and stress components are

$$\overline{\underline{e'}}_6 = [\omega]^6_6 \, \overline{\underline{e}}_6 \,, \quad \overline{\underline{E'}}_6 = [\omega]^6_6 \, \overline{\underline{E}}_6 \,. \qquad \dots (9)$$

From (7) and (9) it follows that

$$\overline{\underline{E'}}_6 = [\omega]^6_6 \, [c]^6_6 \, \overline{\underline{\Omega}}{}^6_6 \cdot \overline{\underline{e}}_6 \,, \quad \textit{i.e.} \ [c']^6_6 = [\omega]^6_6 \, [c]^6_6 \, \overline{\underline{\Omega}}{}^6_6 \,.$$

Thus the body is isotropic if and only if we always have

$$[\omega]_6^6\ [c]_6^6\ \overline{\underline{\Omega}}_6^6=[c]_6^6 \qquad \dots \quad (10)$$

i.e.

$$[\omega]_6^6\ [c]_6^6=[c]_6^6\ [\omega]_6^6 \qquad \dots \quad (10')$$

however the axes (OX′, OY′, OZ′) are chosen.

First let the new axes be formed by merely reversing the axis of x. Then we have

$$|e_1', e_2', e_3', e_4', e_5', e_6'|=|e_1, e_2, e_3, e_4, -e_5, -e_6|.$$

$$[E_1', E_2', E_3', E_4', E_5', E_6']=[E_1, E_2, E_3, E_4, -E_5, -E_6].$$

In this case $[c']_6^6$ is formed by changing the signs of the first four elements in the last two vertical rows and the last two horizontal rows of $[c]_6^6$, and the equation $[c']_6^6 = [c]_6^6$ requires that all those 16 elements shall vanish. From this result and the similar results obtained by reversing the axes of y and z, we see that it is a necessary condition for isotropy that $[c]_6^6$ must have the form

$$[c]_6^6=\begin{bmatrix} c, & 0 \\ 0, & k \end{bmatrix}_{3,3}^{3,3}, \text{ where } [k]_3^3=\begin{vmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{vmatrix}. \qquad \dots \quad (11)$$

If we give it this form, and write

$$[\omega]_6^6=\begin{bmatrix} u, & p \\ q, & r \end{bmatrix}_{3,3}^{3,3}. \qquad \dots \quad (12)$$

the necessary and sufficient conditions (10′) can be replaced by

$$[u]_3^3[c]_3^3=[c]_3^3[u]_3^3, \qquad \dots \quad (13)$$

$$[v]_3^3[k]_3^3=[k]_3^3[v]_3^3, \qquad \dots \quad (14)$$

$$[p]^3_3[k]^3_3=[c]^3_3[p]^3_3, \qquad \dots \quad (15)$$

$$[q]^3_3[c]^3_3=[k]^3_3[q]^3_3, \qquad \dots \quad (16)$$

where $[u]^3_3$, $[v]^3_3$, $[p]^3_3$, $[q]^3_3$ have the values shown by (8). From (14) it follows at once that another necessary condition is $k_1=k_2=k_3$; and we can therefore write

$$k_1=k_2=k_3=\mu.$$

Next let the new axes (OX′, OY′, OZ′) be formed by turning the axes (OX, OY, OZ) through any angle θ about OZ, so that

$$\begin{bmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{bmatrix} = \begin{bmatrix} \cos\theta, & \sin\theta, & 0 \\ -\sin\theta, & \cos\theta, & 0 \\ 0, & 0, & 1 \end{bmatrix}$$

Then the necessary equation (13) is

$$\begin{bmatrix} \cos^2\theta, & \sin^2\theta, & 0 \\ \sin^2\theta, & \cos^2\theta, & 0 \\ 0, & 0, & 1 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \cos^2\theta, & \sin^2\theta, & 0 \\ \sin^2\theta, & \cos^2\theta, & 0 \\ 0, & 0, & 1 \end{bmatrix},$$

and this equation is satisfied for all values of θ if and only if

$$c_{21}=c_{12}, \quad c_{31}=c_{32}, \quad c_{11}=c_{22}.$$

From this result and the corresponding results obtained from rotations about OX, OY, we see that it must be possible to write

$$c_{23}=c_{32}=c_{31}=c_{13}=c_{12}=c_{21}=\lambda, \quad c_{11}=c_{22}=c_{33}=\lambda+\rho,$$

and that the condition (13) is then always satisfied.

Finally the conditions (15) and (16) are satisfied when and only when

$$\rho=\mu.$$

Thus the necessary and sufficient condition for isotropy in the unstrained state is that $[c]_6^6$ shall have the form

$$[c]_6^6=\begin{vmatrix} \lambda+\mu, & \lambda, & \lambda, & 0, & 0, & 0 \\ \lambda, & \lambda+\mu, & \lambda, & 0, & 0, & 0 \\ \lambda, & \lambda, & \lambda+\mu, & 0, & 0, & 0 \\ 0, & 0, & 0, & \mu, & 0, & 0 \\ 0, & 0, & 0, & 0, & \mu, & 0 \\ 0, & 0, & 0, & 0, & 0, & \mu \end{vmatrix}.$$

When we replace μ by 2μ, and write $e_{xx}+e_{yy}+e_{zz}=\Delta$, the corresponding stress-strain relations are

$$X_x=\lambda\Delta+2\mu\, e_{xx}, \quad Y_y=\lambda\Delta+2\mu\, e_{yy}, \quad Z_z=\lambda\Delta+2\mu\, e_{zz}.$$

$$Y_z=\mu e_{yz}, \qquad Z_x=\mu e_{zx}, \qquad X_y=\mu e_{xy}.$$

5. *The stress-strain relations for any isotropic body.*

If, as must be the case when the body is not a solid, the unstrained state is not one of zero stress, we must replace (7) by

$$\begin{bmatrix} E \\ 1 \end{bmatrix}_{6,1} = \begin{vmatrix} c, & d \\ 0, & 1 \end{vmatrix}_{6,1}^{6,1} \begin{bmatrix} e \\ 1 \end{bmatrix}_{6,1}, \qquad \ldots \quad (18)$$

and (10′) by

$$\begin{bmatrix}\omega, & 0\\ 0, & 1\end{bmatrix}_{6,1}^{6,1}\begin{bmatrix}c, & d\\ 0, & 1\end{bmatrix}_{6,1}^{6,1} = \begin{bmatrix}c, & d\\ 0, & 1\end{bmatrix}_{6,1}^{6,1}\begin{bmatrix}\omega, & 0\\ 0, & 1\end{bmatrix}_{6,1}^{6,1}. \qquad \ldots \quad (19)$$

The conditions that the body shall be isotropic in the unstrained state are the same as before together with the additional condition

$$[\omega]_6^6\,[d]_6^1 = [d]_6^1 \qquad \ldots \quad (20)$$

Considering the three particular cases in which the direction of one of the three axes of co-ordinates is merely reversed, we see from (20) that we must have

$$d_{41} = d_{51} = d_{61} = 0, \qquad \ldots \quad (21)$$

and these values reduce (20) to

$$[u]_3^3\,[d]_3^1 = [d]_3^1,\; 0 = [q]_3^3\,[d]_3^1 \qquad \ldots \quad (22)$$

Considering the cases of rotations about OX, OY, OZ, we see that the first of the conditions (22) can only be satisfied when we can write

$$d_{11} = d_{21} = d_{31} = -p,$$

and then both conditions are always satisfied.

Thus any slightly strained body whatever is isotropic in the unstrained state if and only if the stress-strain relations are

$$\begin{vmatrix} X_x \\ Y_y \\ Z_z \\ Y_z \\ Z_x \\ X_y \end{vmatrix} = \begin{vmatrix} \lambda+2\mu, & \lambda, & \lambda, & 0, & 0, & 0, & -p \\ \lambda, & \lambda+2\mu, & \lambda, & 0, & 0, & 0, & -p \\ \lambda, & \lambda, & \lambda+2\mu, & 0, & 0, & 0, & -p \\ 0, & 0, & 0, & \mu, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & \mu, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & \mu, & 0 \end{vmatrix} \begin{vmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{yz} \\ e_{zx} \\ e_{xy} \\ 1 \end{vmatrix},$$

or

$$X_x = \lambda\Delta + 2\mu\, e_{xx} - p, \quad Y_y = \lambda\Delta + 2\mu\, e_{yy} - p, \quad Z_z = \lambda\Delta + 2\mu\, e_{zz} - p,$$

$$Y_z = \mu\, e_{yz}, \qquad Z_x = \mu\, e_{zx}, \qquad X_y = \mu\, e_{xy}.$$

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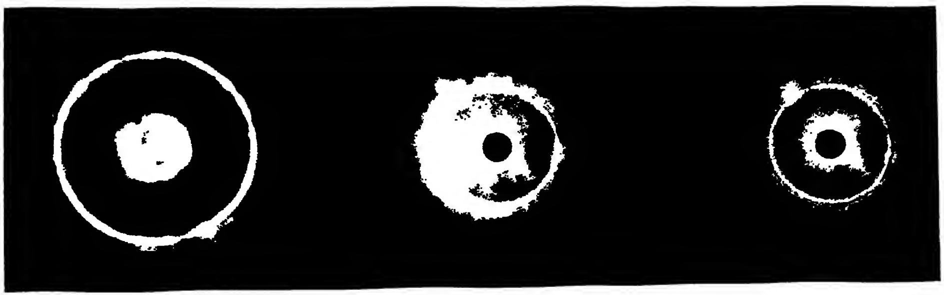
Fig. 1 Fig. 2 Fig. 3



Illustrating the formation of optical images by a diffracting boundary

ON THE FORMATION OF OPTICAL IMAGES BY A DIFFRACTING BOUNDARY.

[*With a Plate*]

BY

BHUPENDRA CHANDRA DAS.

In a recent paper[1] published in the Philosophical Magazine, Prof. Banerji has noticed that if a circular aperture placed in front of a lens is illuminated by a point source of light and if a small screen is placed in the focal plane so as to cut off the entire geometrical cone of rays, a bright image of the source may be traced along the axis behind the screen and for a considerable distance beyond. As remarked by him, this phenomenon is somewhat analogous to that observed by Porter[2] and Hufford[3] almost simultaneously, namely, that, the rays diffracted by a circular disk can form an optical image of the source along the axis of symmetry, but differs from it, as in this case the image is formed by the rays diffracted by the boundary of a circular aperture. At the suggestion of Prof. Banerji I undertook a detailed study of this phenomenon and have succeeded in obtaining a mathematical theory. While carrying out the experiment, I have observed that the central bright spot is surrounded by two sets of alternately bright and dark rings, the outer set being at a considerable distance apart from the inner one. A large number of very faint rings may also be observed in the space between the first set of rings and the second, but not between the first set of rings and the bright spot, which region is marked by almost complete darkness. It appears that the configuration of the outer set of rings depends to a considerable extent on the form of the screen placed

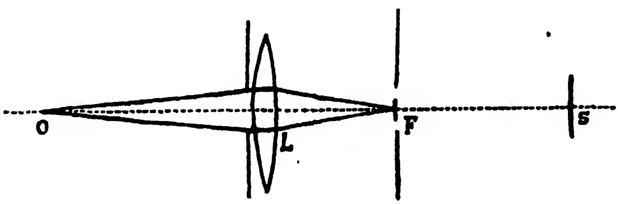
[1] Banerji, "On the radiation of light from the boundaries of diffracting apertures," *Phil. Mag.*, vol. xxxvii., Jan. 1919.

[2] Porter, "On the formation of images by means of an opaque disk," *Phil. Mag.*, vol. xxvii, p. 673 (1914).

[3] Hufford, "Some new diffraction photographs," *Phys. Rev.* vol. III, Ser. 2, p 241 (1914).

in the focal plane. These rings undergo modification if the screen in the focal plane be either displaced or replaced by one having a different boundary. Figures I and II in the plate illustrate the phenomenon observed with a circular diffracting aperture illuminated by a strong beam of light allowed to pass through a circular pin hole. The central bright spot has a circular shape in these photographs. The formation of optical images by a circular diffracting boundary illuminated by a non-circular point source of light is illustrated in figure III. In this photograph the source is triangular and the central bright spot will be noticed to have also a triangular shape. In any case the image is found to closely follow the form of the source.

Let r be the radius of the circular aperture of a lens L which is illuminated by a point source of light O and let the beam converge to a focus F at a distance b from the aperture. Let the light at the focal plane be cut off by means of an opaque circular disk of small radius ζ_1. For the sake of giving definiteness to the problem, we shall impose an upper limit to ζ, say ζ_2, that is to say, we shall suppose the diffracted rays to pass through an annular opening in the focal plane.



The disturbance at any point P (ζ, ϕ), in the focal plane is known to be

$$K \frac{J_1\left(\frac{mr}{b}\zeta\right)}{\cdot} \sin nt,$$

where K is a constant and $m = \frac{2\pi}{\lambda}$, λ being the wave length of light.

The disturbance at any point Q, distant x from the axis, on a screen S placed at any distance c behind the focal plane can be regarded as due to the diffracted rays which pass through the annulus.

Now if R be the distance of Q from P, then

$$R^2 = (\zeta \cos \phi - x)^2 + \zeta^2 \sin^2 \phi + c^2$$

$$= \zeta^2 - 2\zeta x \cos \phi + x^2 + c^2.$$

The effect at Q due to an elementary disturbance at P is, by Huyghen's principle, equal to

$$\zeta d\zeta\, d\phi\ \mathrm{K} \frac{\mathrm{J}_1\left(\frac{mr}{b}\zeta\right)}{\zeta} \cdot \frac{1}{\mathrm{R}} \sin(m\mathrm{R}-nt).$$

Hence the total disturbance at Q is given by

$$\Psi=\mathrm{K}\int_0^{2\pi}\int_{\zeta_1}^{\zeta_2} \frac{1}{\mathrm{R}}\, \mathrm{J}_1\left(\frac{mr}{b}\zeta\right) \sin(m\mathrm{R}-nt)\ d\zeta d\phi.$$

If the opening be small, and Q is very near the axis we may substitute $\frac{1}{c}$ for $\frac{1}{\mathrm{R}}$, and the expression for the total disturbance becomes

$$\Psi=\frac{\mathrm{K}}{c}\int_0^{2\pi}\int_{\zeta_1}^{\zeta_2} \mathrm{J}_1\left(\frac{mr}{b}\zeta\right) \sin(m\mathrm{R}-nt)\ d\zeta d\phi.$$

Since x and ζ are very small campared to b or c, we have, on extracting the square root and neglecting terms of the higher order,

$$\mathrm{R}=c+\frac{\zeta^2}{2c}+\frac{x^2}{2c}-\frac{\zeta x}{c}\cos\phi.$$

Putting $m\left(c+\frac{x^2}{2c}\right)-nt=\omega$, we can write

$$\Psi=\frac{\mathrm{K}}{c}\left[\mathrm{C}\sin\omega+\mathrm{S}\cos\omega\right], \qquad \ldots \quad (\mathrm{i})$$

where

$$\mathrm{C}=\int_0^{2\pi}\int_{\zeta_1}^{\zeta_2} \mathrm{J}_1\left(\frac{mr}{b}\zeta\right) \cos\left(\frac{m\zeta^2}{2c}-\frac{mx}{c}\zeta\cos\phi\right) d\zeta d\phi.$$

$$\mathrm{S}=\int_0^{2\pi}\int_{\zeta_1}^{\zeta_2} \mathrm{J}_1\left(\frac{mr}{b}\zeta\right) \sin\left(\frac{m\zeta^2}{2c}-\frac{mx}{c}\zeta\cos\phi\right) d\zeta d\phi.$$

On integrating with respect to ϕ we easily get,

$$C=2\pi\int_{\zeta_1}^{\zeta_2} J_1(A\zeta)\,J_0(B\zeta)\cos\left(\frac{1}{2}\mu\zeta^2\right)d\zeta, \qquad \ldots \quad \text{(ii)}$$

$$S=2\pi\int_{\zeta_1}^{\zeta_2} J_1(A\zeta)\,J_0(B\zeta)\sin\left(\frac{1}{2}\mu\zeta^2\right)d\zeta, \qquad \ldots \quad \text{(iii)}$$

where

$$A=\frac{mr}{b}, \; B=\mu x, \; \mu=\frac{m}{c}.$$

We now proceed to evaluate these integrals. Integrating by parts, without putting in the limits for the present, we obtain

$$\int J_1(A\zeta)\,J_0(B\zeta)\cos\left(\frac{1}{2}\mu\zeta^2\right)d\zeta=\cos\left(\frac{1}{2}\mu\zeta^2\right)\int J_1(A\zeta)\,J_0(B\zeta)\,d\zeta$$

$$+\mu\int\zeta\sin\left(\frac{1}{2}\mu\zeta^2\right)\int J_1(A\zeta)\,J_0(B\zeta)\,d\zeta\,d\zeta. \qquad \ldots \quad \text{(iv)}$$

Now remembering the formulae

$$\left.\begin{aligned} &\int x^n J_{n-1}(x)\,dx=x^n J_n(x), \\ &\frac{d}{dx}\left[\frac{J_n(x)}{x^n}\right]=-\frac{J_{n+1}(x)}{x^n}, \end{aligned}\right\} \qquad \ldots \quad \text{(v)}$$

we get

$$\int J_1(A\zeta)\,J_0(B\zeta)\,d\zeta=\frac{A}{B^2}\int\frac{J_1(A\zeta)}{A\zeta}.B\zeta\,J_0(B\zeta)\,d(B\zeta)$$

$$=\frac{A}{B^2}\cdot\frac{J_1(A\zeta)}{A\zeta}.B\zeta\,J_1(B\zeta)+\frac{A^2}{B^2}\int\frac{J_2(A\zeta)}{A\zeta}.B\zeta\,J_1(B\zeta)\,d\zeta$$

$$=\frac{1}{A}\cdot\frac{A}{B}J_1(A\zeta)\,J_1(B\zeta)+\frac{A^3}{B^4}\int\frac{J_2(A\zeta)}{(A\zeta)^2}.(B\zeta)^2 J_1(B\zeta)\,d(B\zeta)$$

and by successive application of (v) this becomes finally

$$= \frac{1}{A}\left[\frac{A}{B} J_1 (A\zeta) J_1 (B\zeta) + \frac{A^2}{B^2} J_2 (A\zeta) J_2 (B\zeta)\right.$$

$$\left. + \frac{A^3}{B^3} J_3 (A\zeta) J_3 (B\zeta) + \dots\right].$$

On substitution of the above, the second term of the right hand side of (iv) becomes

$$\frac{\mu}{A}\int \zeta\left[\frac{A}{B} J_1 (A\zeta) J_1 (B\zeta) + \frac{A^2}{B^2} J_2 (A\zeta) J_2 (B\zeta)\right.$$

$$\left. + \frac{A^3}{B^3} J_3 (A\zeta) J_3 (B\zeta) + \dots\right] \sin (\tfrac{1}{2}\mu\zeta^2)\, d\zeta$$

$$= \frac{\mu}{A} \sin (\tfrac{1}{2}\mu\zeta^2) \int \zeta\left[\frac{A}{B} J_1 (A\zeta) J_1 (B\zeta) + \frac{A^2}{B^2} J_2 (A\zeta) J_2 (B\zeta)\right.$$

$$\left. + \dots\right] d\zeta$$

$$+\frac{\mu^2}{A}\int \zeta \cos (\tfrac{1}{2}\mu\zeta^2) \int \zeta\left[\frac{A}{B} J_1 (A\zeta) J_1 (B\zeta)\right.$$

$$\left. \frac{A^2}{B^2} J_2 (A\zeta) J_2 (B\zeta) + \dots\right] d\zeta . d\zeta.$$

Again, proceeding as in the previous case,

$$\int \zeta \frac{A}{B} J_1 (A\zeta) J_1 (B\zeta)\, d\zeta = \frac{\zeta}{B}\left[\frac{A}{B} J_1 (A\zeta) J_2 (B\zeta)\right.$$

$$\left. + \frac{A^2}{B^2} J_2 (A\zeta) J_3 (B\zeta) + \dots\right],$$

$$\int \zeta \frac{A^2}{B^2} J_2 (A\zeta) J_2 (B\zeta)\, d\zeta = \frac{\zeta}{B}\left[\frac{A^2}{B^2} J_2 (A\zeta) J_3 (B\zeta) + \dots\right]$$

Hence

$$\int \zeta \left[\frac{A}{B} J_1(A\zeta) J_1(B\zeta) + \frac{A^2}{B^2} J_2(A\zeta) J_2(B\zeta) + \dots \right] d\zeta$$

$$= \frac{\zeta}{B} \left[\frac{A}{B} J_1(A\zeta) J_2(B\zeta) + 2\frac{A^2}{B^2} J_2(A\zeta) J_3(B\zeta) \right.$$

$$\left. + 3\frac{A^3}{B^3} J_3(A\zeta) J_4(B\zeta) + \dots \right]$$

Proceeding to perform the integrations in this way we finally arrive at the result,

$$\int J_1(A\zeta) J_0(B\zeta) \cos(\tfrac{1}{2}\mu\zeta^2)\, d\zeta = \frac{1}{A} \cos(\tfrac{1}{2}\mu\zeta^2) \left[D_0 - \left(\frac{\mu\zeta}{B}\right)^2 D_2 \right.$$

$$\left. + \left(\frac{\mu\zeta}{B}\right)^4 D_4 - \dots \right] + \frac{1}{A} \sin(\tfrac{1}{2}\mu\zeta^2) \left[\left(\frac{\mu\zeta}{B}\right) D_1 - \left(\frac{\mu\zeta}{B}\right)^3 D_3 + \dots \right].$$

where

$$D_0 = \frac{A}{B} J_1(A\zeta) J_1(B\zeta) + \frac{A^2}{B^2} J_2(A\zeta) J_2(B\zeta)$$

$$+ \frac{A^3}{B^3} J_3(A\zeta) J_3(B\zeta) + .$$

$$D_1 = \frac{A}{B} J_1(A\zeta) J_2(B\zeta) + 2\frac{A^2}{B^2} J_2(A\zeta) J_3(B\zeta)$$

$$+ 3\frac{A^3}{B^3} J_3(A\zeta) J_4(B\zeta) + \dots$$

$$D_2 = \frac{A}{B} J_1(A\zeta) J_3(B\zeta) + 3\frac{A^2}{B^2} J_2(A\zeta) J_4(B\zeta)$$

$$+ 6\frac{A^3}{B^3} J_3(A\zeta) J_5(B\zeta) + \dots$$

If we put

$$M = D_0 - \left(\frac{\mu\zeta}{B}\right)^2 D_2 + \left(\frac{\mu\zeta}{B}\right)^4 D_4 - ..$$

$$N = \left(\frac{\mu\zeta}{B}\right) D_1 - \left(\frac{\mu\zeta}{B}\right)^3 D_3 + \ldots$$

we can write,

$$C = \frac{2\pi}{A}\left[M\cos(\tfrac{1}{2}\mu\zeta^2) + N\sin(\tfrac{1}{2}\mu\zeta^2)\right]_{\zeta_1}^{\zeta_2}$$

and exactly in the same way,

$$S = \frac{2\pi}{A}\left[M\sin(\tfrac{1}{2}\mu\zeta^2) - N\cos(\tfrac{1}{2}\mu\zeta^2)\right]_{\zeta_1}^{\zeta_2}$$

The terms here are expressed in ascending powers of $\frac{A}{B}$ and $\frac{\mu}{B}$ and the results therefore hold good only when these are less than unity, in which case the series are all convergent. Near the axis however, where we require our results to hold good, r is very small and so also is B and thus $\frac{A}{B}$ is very large. We shall in these cases have to express our results in a series of $\frac{B}{A}$. It is easy to do this by the previous method, provided we note that in evaluating the integrals like $\int J_1(A\zeta)J_0(B\zeta)d\zeta$ by parts, we shall have to integrate $J_1(A\zeta)$ and differentiate $J_0(B\zeta)$ instead of the reverse method adopted in the previous case. The final result can be written in the form

$$C = \frac{2\pi}{A}\left[-P\cos(\tfrac{1}{2}\mu\zeta^2) - Q\sin(\tfrac{1}{2}\mu\zeta^2)\right]_{\zeta_1}^{\zeta_2},$$

$$S = \frac{2\pi}{A}\left[-P\sin(\tfrac{1}{2}\mu\zeta^2) + Q\cos(\tfrac{1}{2}\mu\zeta^2)\right]_{\zeta_1}^{\zeta_2}$$

where

$$P = E_0 - \left(\frac{\mu\zeta}{A}\right)^2 E_2 + \left(\frac{\mu\zeta}{A}\right)^4 E_4 - \ldots$$

$$Q = \left(\frac{\mu\zeta}{A}\right) E_1 - \left(\frac{\mu\zeta}{A}\right)^3 E_3 + .$$

and

$$E_0 = J_0(A\zeta)J_0(B\zeta) + \frac{B}{A} J_1(A\zeta)\, J_1(B\zeta) + \frac{B^2}{A^2} J_2(A\zeta)J_2(B\zeta) + \ldots,$$

$$E_1 = J_1(A\zeta)\, J_0(B\zeta) + 2\frac{B}{A} J_2(A\zeta)\, J_1(B\zeta) + 3\frac{B^2}{A^2} J_3(A\zeta)J_2(B\zeta) + \ldots,$$

$$E_2 = J_2(A\zeta)\, J_0(B\zeta) + 3\frac{B}{A} J_3(A\zeta)J_1(B\zeta) + 6\frac{B^2}{A^2} J_4(A\zeta)J_2(B\zeta) + \ldots,$$

etc.

If I denote the intensity of illumination, then

$$I = C^2 + S^2.$$

The only quantity involving x is B which is $=\mu x$, and to discuss the maxima or minima on the axis where $x=0$, we are to remember that,

$$\underset{x\to 0}{\mathrm{Lt.}}\; J_0(x) = 1, \quad \underset{x\to 0}{J_1(x)} = \underset{x\to 0}{J_2(x)} = \ldots = 0.$$

We now easily get,

$$\underset{x\to 0}{E_0} = J_0(A\zeta), \quad \underset{x\to 0}{E_1} = J_1(A\zeta), \quad \underset{x\to 0}{E_2} = J_2(A\zeta), \ldots \text{etc.} \ldots$$

$$\underset{x\to 0}{\frac{\partial E_0}{\partial x}} = \underset{x\to 0}{\frac{\partial E_1}{\partial x}} = \underset{x\to 0}{\frac{\partial E_2}{\partial x}} = \ldots = 0.$$

Also since

$$\underset{x\to 0}{J_1'}(x) = \tfrac{1}{2},$$

we get on a second differentiation and substitution of $x=0$,

$$\underset{x\to 0}{\mathrm{Lt.}}\; \frac{\partial^2 E_0}{\partial x^2} = -\tfrac{1}{2}\mu^2\zeta^2\, J_0(A\zeta) = -\tfrac{1}{2}\mu^2\zeta^2 \underset{x\to 0}{E_0},$$

$$\underset{x\to 0}{\mathrm{Lt.}}\; \frac{\partial^2 E_1}{\partial x^2} = -\tfrac{1}{2}\mu^2\zeta^2 \underset{x\to 0}{E_1},$$

$$\underset{x\to 0}{\mathrm{Lt.}}\; \frac{\partial^2 E_2}{\partial x^2} = -\tfrac{1}{2}\mu^2\zeta^2 \underset{x\to 0}{E_2},$$

... ...

Thus

$$\underset{x\to 0}{\mathrm{Lt.}}\, P = J_0(A\zeta) - \left(\frac{\mu\zeta}{A}\right)^2 J_2(A\zeta) + \left(\frac{\mu\zeta}{A}\right)^4 J_4(A\zeta) - \dots,$$

$$\underset{x\to 0}{\mathrm{Lt.}}\, Q = \left(\frac{\mu\zeta}{A}\right) J_1(A\zeta) - \left(\frac{\mu\zeta}{A}\right)^3 J_3(A\zeta) + \dots,$$

$$\underset{x\to 0}{\mathrm{Lt.}}\, \frac{\partial P}{\partial x} = \underset{x\to 0}{\mathrm{Lt.}}\, \frac{\partial Q}{\partial x} = 0,$$

$$\underset{x\to 0}{\mathrm{Lt.}}\, \frac{\partial^2 P}{\partial x^2} = -\tfrac{1}{2}\mu^2\zeta^2 \underset{x\to 0}{P}, \quad \underset{x\to 0}{\mathrm{Lt.}}\, \frac{\partial^2 Q}{\partial x^2} = -\tfrac{1}{2}\mu^2\zeta^2 \underset{x\to 0}{Q}.$$

Denoting the values of

$$\frac{2\pi}{A}\left[-P\cos(\tfrac{1}{2}\mu\zeta^2) - Q\sin(\tfrac{1}{2}\mu\zeta^2)\right]$$

by C_1 and C_2 when ζ_1 and ζ_2 respectively are substituted for ζ, and the corresponding values of

$$\frac{2\pi}{A}\left[-P\sin(\tfrac{1}{2}\mu\zeta^2) + Q\cos(\tfrac{1}{2}\mu\zeta^2)\right]$$

by S_1 and S_2 we get

$$C = C_2 - C_1, \quad S = S_2 - S_1,$$

$$\underset{x\to 0}{\mathrm{Lt.}}\, \frac{\partial C}{\partial x} = \underset{x\to 0}{\mathrm{Lt.}}\, \frac{\partial S}{\partial x} = 0,$$

$$\underset{x\to 0}{\mathrm{Lt.}}\, \frac{\partial^2 C}{\partial x^2} = -\tfrac{1}{2}\mu^2\zeta_2{}^2 C_2 + \tfrac{1}{2}\mu^2\zeta_1{}^2 C_1$$

$$\underset{x\to 0}{\mathrm{Lt.}}\, \frac{\partial^2 S}{\partial x^2} = -\tfrac{1}{2}\mu^2\zeta_2{}^2 S_2 + \tfrac{1}{2}\mu^2\zeta_1{}^2 S_1$$

Hence

$$\underset{x\to 0}{\mathrm{Lt.}}\, \frac{\partial I}{\partial x} = \underset{x\to 0}{\mathrm{Lt.}}\, \frac{1}{2}\left\{C\frac{\partial C}{\partial x} + S\frac{\partial S}{\partial x}\right\} = 0,$$

$$\underset{x\to 0}{\mathrm{Lt.}}\, \frac{\partial^2 I}{\partial x^2} = \underset{x\to 0}{\mathrm{Lt.}}\, \frac{1}{2}\left\{C\frac{\partial^2 C}{\partial x^2} + S\frac{\partial^2 S}{\partial x^2} + \left(\frac{\partial C}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial x}\right)^2\right\}$$

$$= \underset{x\to 0}{\mathrm{Lt.}}\, -\tfrac{1}{4}\mu^2\Big[(C_2 - C_1)(\zeta_2{}^2 C_2 - \zeta_1{}^2 C_1)$$

$$+ (S_2 - S_1)(\zeta_2{}^2 S_2 - \zeta_1{}^2 S_1)\Big].$$

The intensity is maximum for all points on the axis provided $\frac{d^2 I}{dx^2}$, when $x=0$, is negative. It will appear from the above expression that this quantity is negative when $\zeta_2=\zeta_1$ and will continue to remain negative even when ζ_2 and ζ_1 differ from each other. The exact stage at which the expression may change from negative to a positive value will depend on the actual magnitudes of C_2, C_1 and S_2, S_1. This gives a qualitative explanation of the formation of images behind the screen.

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On Joachimsthal's Attraction Problem

BY

Sasindra Chandra Dhar

1. Given the total attraction at any point due to an infinite homogeneous rod as an arbitrary function of the perpendicular distance of the point, Joachimsthal's problem [1] consists in determining the law of attraction. Assuming that the law of force is $\psi(r)$, Joachimsthal succeeded in solving the problem for the case when $\psi(r)$ is expansible in a series of negative powers of r. In the present paper, I have shown that the problem can be solved without making this assumption. I have also given the solution of an allied problem which involves the determination of the law of density in an infinite rod, when the total attraction at any point is given as an arbitrary function of the perpendicular distance of the point from the rod, the law of force being the Newtonian Law.

2. The problem as given by Joachimsthal [2] is the following :—

The elements of an infinitely long and homogeneous rod AB attracts the point O whose distance from the rod is h, according to an unknown function of r, *i.e.*, $\psi(r)$. To find this function, if by observation, the total attraction along the perpendicular OM is known to be $F(h)$, where $OM=h$.

If, for the sake of simplicity, the mass of the particle O, the density of the rod and the constant of attraction be taken equal to unity, then the attraction along OP due to an element dt at P on the rod ($MP=t$) is

$$\psi(r)\,dt,$$

and its resolved part along OM is

$$\psi(r)\cos\theta\,dt,$$

θ being the angle MOP.

[1] Joachimsthal : *Crelle's Journal*. Bd. 58.

[2] My attention was drawn to this problem by Dr. Ganesh Prasad, to whom I beg to express my obligations.

Joachimsthal ; *loc-cit.*

Therefore, the total attraction along OM due to the infinite rod AB is

$$2h\int_h^\infty \frac{\psi(r)\,dr}{\sqrt{r^2-h^2}}\,.$$

If we write $f(h)$ for $F(h)/2h$, we are required to solve the following integral equation of the first kind :—

$$f(h)=\int_h^\infty \frac{\psi(r)\,dr}{\sqrt{r^2-h^2}}\,. \qquad \dots \quad (1)$$

The solution which Professor Joachimsthal has obtained is based on the assumption that $\psi(r)$, the unknown function, is capable of expansion in negative integral powers of r. It is possible to obtain the solution of the integral equation (1) without making this assumption.

3. Examining the form of the solution of the integral equation (1) as obtained by Joachimsthal, we may assume, (following a method of Goursat),[1] that

$$\psi(r)=\int_r^\infty \frac{r\phi(x)\,dx}{\sqrt{x^2-r^2}}\,, \qquad \dots \quad (2)$$

where $\phi(x)$ is the unknown expression whose form is to be determined in such a way that the equation (1) is satisfied.

If we put $r=hr'$, we get from (1) and (2)

$$f(h)=\int_1^\infty \frac{\psi(hr')\,dr'}{\sqrt{r'^2-1}}\,,$$

$$\psi(hr')=\int_{hr'}^\infty \frac{hr'\phi(x)dx}{\sqrt{x^2-h^2r'^2}}\,.$$

1. E. Goursat "Sur un probleme D'inversion Resolue Par Abel," *Acta Mathematica*, Vol. 27, 1903.

Hence in the latter integral, if we put $x = hx'$, we get

$$\psi(hr') = h \int_{x'}^{\infty} \frac{r' \phi(hx')\, dx'}{\sqrt{x'^2 - r'^2}}.$$

Therefore,

$$f(h) = h \int_{1}^{\infty} \frac{r' dr'}{\sqrt{r'^2 - 1}} \int_{r'}^{\infty} \frac{\phi(hx')\, dx'}{\sqrt{x'^2 - r'^2}},$$

which, on changing the order of integration (by Dirichlet's formula) becomes

$$f(h) = h \int_{1}^{\infty} \phi(hx')\, dx' \int_{1}^{x'} \frac{r' dr'}{\sqrt{(r'^2 - 1)(x'^2 - r'^2)}}$$

$$= \frac{h\pi}{2} \int_{1}^{\infty} \phi(hx')\, dx' = \frac{\pi}{2} \int_{h}^{\infty} \phi(x)\, dx$$

whence

$$\phi(x) = \frac{-2}{\pi} f'(x). \qquad \ldots \quad (3)$$

Thus the form of $\phi(x)$ is determined and the solution of the integral equation (1) is obtained in the form

$$\psi(r) = \frac{-2}{\pi} \int_{r}^{\infty} \frac{r f'(x)\, dx}{\sqrt{x^2 - r^2}} = \frac{-2r}{\pi} \int_{r}^{\infty} \frac{\frac{d}{dx}\left(\frac{F(x)}{x}\right)}{\sqrt{x^2 - r^2}}. \qquad \ldots \quad (4)$$

It should be noted that the integral equation (1) has a solution, only if $f(h) = 0$, when $h = \infty$. When this condition is satisfied, its solution is given by (4).

4. The solution of the integral equation (1) may also be obtained in series, by following the method of Volterra. By a suitable transformation it is possible to write (1) in the form

$$\psi(z) = -\frac{\sqrt{2z}}{\pi} \phi'(z) + \frac{\sqrt{2z}}{\pi} \int_{z}^{\infty} K_1(z, r)\, \psi(r)\, dr, \qquad \ldots \quad (5)$$

where

$$\phi(z)=\int_z^{\infty}\frac{f(x)\,dx}{\sqrt{z-x}},$$

$$K(z,r)=\int_z^{r}\frac{dx}{\sqrt{z-x}\,\sqrt{r^2-x^2}},\quad (r>z)$$

and
$$K_1(z,r)=\frac{\partial K}{\partial z}$$

The equation (5) is the well-known form of Volterra's integral equation of the second kind, whose solution in series can be easily obtained.

5. The problem connected with that of Joachimsthal is the following :—

The density of an infinitely long rod AB varies according to a certain law, *viz.*, a function $\psi(t)$ of the distance t from M the foot of the perpendicular from the external point O. To find the function, if by observation, the total attraction of a mass at O, which attracts according to the inverse square of the distance, along the perpendicular OM is known.

We take, for the sake of brevity, the mass at O and the constant of attraction to be unity.

The mass of an element of matter dt at P is $\psi(t)dt$ and hence the attraction along OP due to this mass is $\psi(t)dt/r^2$. Therefore the total attraction along the perpendicular OM is, if we take the density function $\psi(t)$ to be symmetrical about the point M, given by

$$2h\int_0^{\infty}\frac{\psi(t)\,dt}{(h^2+t^2)^{\frac{3}{2}}}\,.$$

If the observed total attraction is given by $f(h)$, then we get the following integral equation of the first kind

$$f(h)=2h\int_0^\infty \frac{\psi(t)\,dt}{(h^2+t^2)^{\frac{3}{2}}}, \qquad \ldots \quad (6)$$

where $\psi(t)$ is the unknown function whose form is required.

6 It should be observed from the study of the integral equation (6) that the necessary condition in order that a solution may exist is that $f(h)=0$ when $h=\infty$, which means physically that there should be no total attraction, when the unit mass O is situated at an infinite distance from the rod. This condition is evidently satisfied in this case.

If we suppose $\psi(t)$ to be of the form

$$\psi(t)=\int_{\sqrt{h^2+t^2}}^\infty \frac{\sqrt{h^2+t^2}.\ x\phi(x)\,dx}{\sqrt{x^2-(h^2+t^2)}}, \qquad \ldots \quad (7)$$

where $\phi(x)$ is an unknown function, whose form is to be determined in such a way that the equation (6) is satisfied. Therefore we get

$$f(h)=2h\int_0^\infty \frac{dt}{(h^2+t^2)^{\frac{3}{2}}}\int_{\sqrt{h^2+t^2}}^\infty \frac{\sqrt{h^2+t^2}.\ x\phi(x)\,dx}{\sqrt{x^2-(h^2+t^2)}}$$

$$=2h\int_h^\infty \frac{dt}{y\sqrt{y^2-h^2}}\int_y^\infty \frac{x\phi(x)\,dx}{\sqrt{x^2-y^2}},$$

(where $y^2=h^2+t^2$)

$$=2h\int_h^\infty x\phi(x)\,dx\int_h^\infty \frac{dy}{y\sqrt{y^2-h^2}\sqrt{x^2-y^2}} \qquad \ldots \quad (8)$$

(on changing the order of integration)

Now, the integral

$$\int_h^x \frac{dy}{y\sqrt{y^2-h^2}\,\sqrt{x^2-y^2}}=\frac{1}{x}\int_{h/x}^1 \frac{dy_1}{y_1\sqrt{(x^2y_1{}^2-h^2)(1-y_1{}^2)}}$$

$$(y=xy_1).$$

$$=\frac{1}{x^2}\int_{h_1}^1 \frac{dy_1}{y_1\sqrt{(y_1{}^2-h_1{}^2)(1-y_1{}^2)}},$$

$$(\text{when } h=xh_1)$$

$$=\frac{-1}{2x^2}\int_{1/h_1{}^2}^1 \frac{d\xi}{\sqrt{(\xi-1)\,(1-h_1{}^2\xi)}}\quad\left(\text{ where } \xi=\frac{1}{y_1{}^2}\right)$$

$$=\frac{-1}{2hx}\int_{\frac{1}{h_1{}^2}}^1 \frac{d\xi}{\sqrt{(\xi-1)\left(\frac{1}{h_1{}^2}-\xi\right)}}$$

If we put

$$\xi=\frac{1}{h_1{}^2}+\left(1-\frac{1}{h_1{}^2}\right)\omega,$$

the above integral becomes equal to

$$\frac{1}{2hx}\int_0^1 \frac{d\omega}{\sqrt{\omega}\sqrt{1-\omega}}=\frac{\pi}{2hx}. \qquad \ldots$$

Hence from (8) we get

$$f(h)=\pi\int_h^\infty \phi(x)\,dx.$$

Therefore

$$\phi(x)=\frac{-1}{\pi}f'(x).$$

Therefore, the solution of the integral equation (6) is given by

$$\psi(t)=-\frac{\sqrt{h^2+t^2}}{\pi}\int_{\sqrt{h^2+t^2}}^\infty \frac{xf'(x)\,dx}{\sqrt{x^2-(h^2+t^2)}}, \qquad \ldots \quad (1$$

which gives the unknown law of density.

On the Potentials of Heterogeneous Incomplete Ellipsoids and Elliptic Discs.

BY

Nikhilranjan Sen

1. The potentials of heterogeneous ellipsoids have engaged the attention of many eminent mathematicians and solutions have been obtained by them in definite integrals and in series of spherical harmonics. These are the two most important forms in which the potential function for an ellipsoid has hitherto been expressed. The density in most cases is taken to be of the form

$$\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)^{\lambda-1} f(x, y, z)$$

and the law of attraction is assumed to be the inverse nth power of the distance. While the case of the complete ellipsoid has thus been studied quite at length by many writers, the potential of an incomplete ellipsoid has not as yet received proper attention. The late Prof. Grube while giving a solution for the potential of a complete ellipsoid indicated a method by which the potential of a uniform incomplete ellipsoid may be determined.[1] The object of this paper is to study the potentials of heterogeneous incomplete ellipsoids and elliptic discs and to notice incidentally the potentials of ellipsoids with certain discontinuous distributions of mass.

The results given here have all been obtained by the use of discontinuous factors, a method first applied by Dirichlet in the determination of the potential of a uniform ellipsoid. The principle is this.[2] The potential is given by the integral

$$V=\iiint\frac{\sigma dv}{r}$$

[1] Crelle, Vols. 69 and 98.

[2] Kronecker, Vorlesungen über bestimmte Integrale.

taken all throughout the volume of the ellipsoid. This integral has been multiplied by Dirichlet by

$$\frac{2}{\pi}\int^{\infty} \frac{\sin\theta \cos s\theta}{\theta} d\theta$$

which has the value unity if $s < 1$ and zero if $s > 1$. If $s=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$ then for all points inside the ellipsoid $s=1$, the integral is unity and for all points outside it is zero.

Hence after multiplication we can take the previous integral throughout the entire infinite space and

$$V = \frac{2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma\frac{\sin\theta \cos s\theta\, d\theta}{\theta}\, \frac{dx\, dy\, dz}{} .$$

This quadruple integral can now be reduced to a single integral.

Later on Dr. Hobson[1] by using a different integral suggested by Kronecker[2] obtained by the application of the same method the potential of a heterogeneous ellipsoid of n dimensions for a more general law of attraction (inverse $(m+1)$th power of the distance).

In the present paper use has been made of Dr. Hobson's integral which is multiplied by a second discontinuous factor chosen according to the condition of the problem and by the application of the method followed by Kronecker and Hobson the potential function is ultimately obtained as a surface integral.

2. We propose to take up the case of the semi-ellipsoid first as it furnishes an important example of the method under consideration. The discontinuous factors to be used now are the following :—

$$(i) \int_{-\infty}^{\infty} \frac{e^{c(q+iw)}}{(q+iw)^{\lambda}}\, dw^{3} = \frac{2\pi}{\Gamma(\lambda)} c^{\lambda-1} \quad \text{if } c>0$$

$$\text{or} \quad 0 \quad \text{if } c<0 \qquad \lambda>1 \quad (\text{q positive})$$

[1] *Proc. Lond. Math. Soc.* vol. 27.

[2] Vorlesungen, *l.c.*

[3] This integral has been used by Hobson, *vide ante.*

$$(ii)\ \frac{1}{\pi}\int_0^{\infty}\frac{\sin\eta v}{v}\,dv+\frac{1}{2} \quad = \quad 1 \quad \text{if } \eta>0$$

$$\text{or} \quad 0 \quad \text{if } \eta<0.$$

Suppose that the semi-ellipsoid (a, b, c) lies on the positive side of the plane $\eta=0$ and is of density

$$\sigma=\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)^{\lambda-1}F\left(\frac{\xi}{a},\frac{\eta}{b},\frac{\zeta}{c}\right) \qquad (\lambda-1\geq 0)$$

$$=\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)^{\lambda-1}e^{\alpha\xi+\beta\eta+\gamma\zeta}F_0\left(\frac{x'}{a},\frac{y'}{b},\frac{z'}{c}\right)$$

where α, β, γ stand for $\frac{\partial}{\partial x'}$, $\frac{\partial}{\partial y'}$, $\frac{\partial}{\partial z'}$ respectively and F_0 means that after the differentiations are performed x', y', z' are all put equal to zero.

Now consider the integral

$$P=\frac{\Gamma(\lambda)}{2\pi}\int\frac{e^{\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)(q+iw)}}{(q+iw)^{\lambda}}$$

$$\times\left[\frac{1}{\pi}\int_0^{\infty}\frac{\sin\eta v}{v}\,dv+\tfrac{1}{2}\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{\alpha\xi+\beta\eta+\gamma\zeta}\,d\xi d\eta d\zeta dw F_0,$$

where

$$r^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2.$$

It is the potential of a semi-ellipsoid (for which $\eta>0$) of density σ under a law of force varying as the inverse $(m+1)$th power of the distance at the point x, y, z. This we write in the form

$$P=\tfrac{1}{2}P_1+V,$$

where P_1 is the potential of the complete ellipsoid of density σ and

$$V=\frac{1}{2\pi^2}\,\Gamma(\lambda)\iiiint\!\!\int\frac{\sin\eta v}{v}\frac{e^{\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)(q+iw)}}{(q+iw)^{\lambda}}\,e^{\alpha\xi+\beta\eta+\gamma\zeta}\,d\xi d\eta d\zeta dv dw\, F_0.$$

It easily follows from the integral (*ii*) which may also be written in the form

$$\frac{2}{\pi}\int_0^\infty \frac{\sin \eta v}{v}\, dv \quad = \quad 1 \qquad \text{if } \eta > 0$$
$$\qquad\qquad\qquad \text{or} \quad -1 \qquad \text{if } \eta < 0,$$

that $2V$ is the potential of an ellipsoid with a discontinuous distribution of mass on opposite sides of the plane $\eta = 0$, *viz.*, of the distribution of a volume density σ throughout the half-ellipsoid for which $\eta > 0$ and of $-\sigma$ throughout the other half given by $\eta < 0$. As the form of σ has been assumed to be such that the potential of the complete ellipsoid for this distribution is known[1] we shall confine our investigation to the determination of the function V only.

3. Now

$$\frac{1}{r^m} = \frac{1}{\Gamma\left(\frac{m}{2}\right)} \int_0^\infty t^{\frac{m}{2}-1}\, e^{-tr^2}$$

Hence

$$V = \frac{1}{2\pi^2 m} \frac{\Gamma(\lambda)}{\Gamma\left(\frac{m}{2}\right)} \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int \frac{\sin \eta v}{v} \frac{e^{v(q+iw)}}{(q+iw)^\lambda}$$

$$e^{\alpha\xi+\beta\eta+\gamma\zeta \;-t\,[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]}$$

$$t^{\frac{m}{2}-1} \quad d\xi\, d\eta\, d\zeta\, dv\, dw\, dt\, F_u.$$

at this stage we integrate with respect to ξ, η, ζ respectively and we have the three following integrals :

$$\int^{\infty} e^{-\left(t+\frac{q+iw}{a^2}\right)\xi^2+2\left(\frac{\alpha}{2}+tx\right)\xi}\, d\xi$$

$$= \frac{\sqrt{\pi}}{\sqrt{t+\frac{q+iw}{a^2}}}\; e^{\left(tx+\frac{\alpha}{2}\right)^2 \Big/ \left(t-\frac{q+iw}{a^2}\right)}$$

[1] Hobson, *l.c.* Dyson, *Quarterly Journal* 1891 vol. xxv.

$$\int^{\infty} e^{-\left(t+\frac{q+iw}{c^2}\right)\zeta^2+2\left(\frac{\gamma}{2}+tz\right)\zeta}\,d\zeta$$

$$= \frac{\sqrt{\pi}}{\sqrt{t+\frac{q+iw}{c^2}}}\, e^{\left(tz+\frac{\gamma}{2}\right)^2 \Big/ \left(t+\frac{q+iw}{c^2}\right)}$$

and

$$\int^{\infty} e^{-\left(t+\frac{q+iw}{b^2}\right)\eta^2+2\left(\frac{\beta}{2}+ty\right)\eta} \sin \eta r\, d\eta = \frac{\sqrt{\pi}}{\sqrt{t+\frac{q+iw}{b^2}}}$$

$$e^{\left[\left(ty+\frac{\beta}{2}\right)^2-\frac{1}{4}r^2\right]\Big/\left(t+\frac{q+iw}{b^2}\right)} \sin\left[\frac{\left(ty+\frac{\beta}{2}\right)r}{t+\frac{q+iw}{b^2}}\right].$$

Hence

$$V = \frac{\Gamma(\lambda)}{2\pi^2 m\Gamma\left(\frac{m}{2}\right)} \iiint \frac{1}{v}\, \frac{e^{(q+iw)}}{(q+iw)^{\lambda}}\, t^{\frac{m}{2}-1}\, e^{-t(x^2+y^2+z^2)}$$

$$\frac{\pi\sqrt{\pi}}{\sqrt{t+\frac{q+iw}{a^2}}\sqrt{t+\frac{q+iw}{b^2}}\sqrt{t+\frac{q+iw}{c^2}}}$$

$$\times e^{\frac{\left(tx+\frac{\alpha}{2}\right)^2}{t+\frac{q+iw}{a^2}}+\frac{\left(ty+\frac{\beta}{2}\right)^2}{t+\frac{q+iw}{b^2}}+\frac{\left(tz+\frac{\gamma}{2}\right)^2}{t+\frac{q+iw}{c^2}}}$$

$$\times e^{-\frac{r^2}{4}\Big/\left(t+\frac{q+iw}{b^2}\right)} \sin\frac{\left(ty+\frac{\beta}{2}\right)r}{t+\frac{q+iw}{b^2}}\, dr\, dw\, dt\, \mathbf{F}_0;$$

now changing the variable from t to θ by the substitution

$$\frac{q+iw}{t} = \theta.$$

we have

$$V=\frac{abc}{2\sqrt{\pi m}}\,\frac{\Gamma(\lambda)}{\Gamma\left(\frac{m}{2}\right)}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\frac{e^{(q+iw)\left(1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right)}}{(q+iw)^{\lambda-\frac{m}{2}+\frac{3}{2}}\,\theta^{\frac{m}{2}-\frac{1}{2}}}$$

$$\frac{a^2\alpha x}{a^2+\theta}+\frac{b^2\beta y}{b^2+\theta}+\frac{c^2\gamma z}{c^2+\theta}\qquad \frac{1}{r}\,\frac{e^{-\frac{1}{4(q+iw)}\left(\frac{a^2\alpha^2\theta}{a^2+\theta}+\frac{b^2\beta^2\theta}{b^2+\theta}+\frac{c^2\gamma^2\theta}{c^2+\theta}\right)}}{\sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)}}$$

$$e^{-\frac{\theta b^2}{4(q+iw)(\theta+b^2)}}\sin r\left[\frac{b^2y}{b^2+\theta}+\frac{1}{2}\,\frac{\beta b^2\theta}{(q+iw)(b^2+\theta)}\right]dw\,d\theta\,dr\ \mathrm{F}.$$

(1) First suppose, $\alpha=\beta=\gamma=0$, *i.e.*, $\sigma=\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)^{\lambda-1}$

then

$$V=\frac{1}{m}\,\frac{abc}{2\sqrt{\pi}}\,\frac{\Gamma(\lambda)}{\Gamma\left(\frac{m}{2}\right)}\int\int\frac{\theta^{\frac{1}{2}-\frac{m}{2}}}{\sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)}}$$

$$\frac{e^{(q+iw)\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right]}}{(q+iw)^{\lambda-\frac{m}{2}+\frac{3}{2}}}\,dw\,d\theta$$

$$\times\int_0^{\infty}e^{-\frac{v^2 b^2\theta}{4(q+iw)(b^2+\theta)}}\,\frac{\sin\frac{vb^2y}{b^2+\theta}}{v}\,dv.$$

Putting $\qquad \frac{b^2y}{b^2+\theta}\,v=u,\quad \frac{dv}{v}=\frac{du}{u}$

and since

$$\int_0^{\infty}e^{-a^2v^2}\cos\beta v\,dv=\frac{\sqrt{\pi}}{2a}\,e^{-\frac{\beta^2}{4a^2}},$$

$$\int_0^{\infty}\int_0^{\beta}e^{-a^2v^2}\cos\beta v\,d\beta\,dv=\frac{\sqrt{\pi}}{2a}\int_0^{\beta}e^{-\frac{\beta^2}{4a^2}}\,d\beta.$$

i.e. $$\int_0^\infty e^{-a^2v^2}\frac{\sin\beta v}{v}\,dv=\frac{\sqrt{\pi}}{2a}\int_0^\infty e^{-\frac{\beta^2}{4a^2}}\,d\beta,$$

and $$\int_0^\infty e^{-a^2v^2}\frac{\sin v}{v}\,dv=\frac{\sqrt{\pi}}{2a}\int_0^1 e^{-\frac{s^2}{4a^2}}\,ds,$$

we have

$$\int_0^\infty e^{-\frac{v^2}{4}\frac{b^2\theta}{(q+iw)(b^2+\theta)}}\sin\frac{b^2y}{b^2+\theta}\,v\,dv$$

$$=\sqrt{\pi}\,\frac{by}{\sqrt{\theta(b^2+\theta)}}(q+iw)^{\frac{1}{2}}\int_0^1 e^{-\frac{b^2y^2}{\theta(b^2+\theta)}(q+iw)s^2}\,ds.$$

Hence

$$V=\frac{1}{m}\,\frac{\Gamma(\lambda)}{2\Gamma\left(\frac{m}{2}\right)}\,ab^2c\,y\int_0^1\int_0^\infty\frac{\theta^{-\frac{m}{2}}\,d\theta\,ds}{(b^2+\theta)\sqrt{(a^2+\theta)(c^2+\theta)}}$$

$$\int_{-\infty}^\infty\frac{e^{(q+iw)\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}\right]}}{(q+iw)^{\lambda-\frac{m}{2}+1}}\,dw.$$

But $$\int_{-\infty}^\infty\frac{e^{(q+iw)\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}\right]}}{(q+iw)^{\lambda-\frac{m}{2}+1}}\,dw$$

$$=\frac{2\pi}{\Gamma\left(\lambda-\frac{m}{2}+1\right)}\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}\right]^{\lambda-\frac{m}{2}},$$

$$\left(\lambda-\frac{m}{2}+1>0\right),$$

or 0 according as

$$1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)},$$

is positive or negative.

Hence

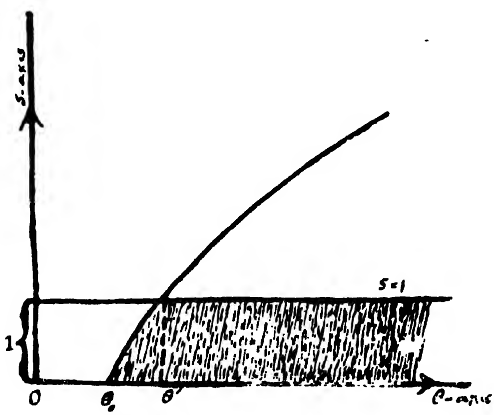
$$V=\frac{1}{m}\,\frac{\Gamma(\lambda)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\lambda-\frac{m}{2}+1\right)}\,\pi a b^2 c y\iint\frac{\theta^{-\frac{m}{2}}}{(b^2+\theta)\sqrt{(a^2+\theta)(c^2+\theta)}}$$

$$\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}\right]^{\lambda-\frac{m}{2}}ds\,d\theta,$$

and the integration is to be taken for those values of s and θ which make

$$R=1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}>0,\ (1\geqq s\geqq 0,\ \theta\text{positive}).$$

4. The above inequality defines the limits to which the integration should be confined. The potential function in this case is expressed as a surface integral and the region of integration wlll be evident from the following considerations :



(i) When the point (x,y,z) is outside the ellipsoid it is evident on writing the inequality in the form

$$R=\left(1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right)-\frac{s^2b^2}{\theta(b^2+\theta)}\,y^2>0$$

that the expression within the bracket in R will be negative so long as θ is less then θ_0 which is the parameter of the confocal ellipsoid passing through the point (x,y,z), *i.e.*, the greatest root of the equation

$$1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}=0.$$

So that no real value of s will make R zero or positive. From the value θ_0 onwards upto infinity the expression within bracket will be positive (its maximum value being 1), hence we can chose s such that R may be positive and for any definite value of θ the maximum value of s that will make R positive is the value of s that makes R vanish. Now it is also obvious from the equation $R=0$ that s which vanishes with $\theta=\theta_0$, will henceforth increase continuously and will approach infinity with θ. If we now draw the line $s=1$. the area of integration is easily found. The point where the curve cuts the line $s=1$ is given by the equation

$$1-\frac{x^2}{a^2+\theta'}-\frac{y^2}{b^2+\theta'}-\frac{z^2}{c^2+\theta'}-\frac{b^2y^2}{\theta'(b^2+\theta')}=0.$$

Hence we can express V as the sum of two such integrals as

$$\int_{\theta_0}^{\theta'}\int_{s=0}^{\frac{\sqrt{\theta(b^2+\theta)}}{by}\left(1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right)^{\frac{1}{2}}} \mathrm{I}\,ds d\theta+\int_{\theta'}^{\infty}\int_{s=0}^{1} \mathrm{I}\,ds d\theta$$

where the integrand is denoted by I.

(*ii*) When the point (x,y,z) is inside the ellipsoid the expression within the bracket will be positive for all values of θ from zero upto infinity and hence we can always find s such that R vanishes. The origin lies on the curve $R=0$ and as θ gradually increases s also continuously increases along with it and ultimately goes to infinity with θ. The region of integration can now be easily obtained by drawing the line $s=1$ and the function V is expressible as the sum of the two integrals

$$\int_{0}^{\theta'}\int_{0}^{\frac{\sqrt{\theta(b^2+\theta)}}{by}\left(1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right)^{\frac{1}{2}}} \mathrm{I}\,ds d\theta+\int_{\theta'}^{\infty}\int_{0}^{1} \mathrm{I}\,ds d\theta.$$

5. The case of the semi-ellipsoid of uniform density under Newtoniam law of attraction is important and may be deduced from

the result given in §3 by putting $\lambda=1$ and $m=1$.

$$V=2ab^2cy\iint\frac{d\theta ds}{(b^2+\theta)\sqrt{\theta(a^2+\theta)(c^2+\theta)}}\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}\right]^{\frac{1}{2}}$$

for the limits defined as before by $R>0$.

This may be reduced to a single integral by integrating according to the scheme of the previous article. For

$$\frac{by}{\sqrt{\theta(b^2+\theta)}}\int\frac{ds}{\sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)}}\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{(b^2+\theta)}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}\right]^{\frac{1}{2}}=\frac{1}{\Delta}\int\beta\sqrt{\alpha^2-\beta^2s^2}\,ds$$

where $\qquad \Delta=\sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)}\ ,\ \beta=\dfrac{by}{\sqrt{\theta(b^2+\theta)}}$

and $\qquad \alpha^2=1-\dfrac{x^2}{a^2+\theta}-\dfrac{y^2}{b^2+\theta}-\dfrac{z^2}{c^2+\theta}$;

this integral as equal to

$$\beta\left[\frac{s}{2}\sqrt{\frac{\alpha^2}{\beta^2}-s^2}+\frac{1}{2}\frac{\alpha^2}{\beta^2}\sin^{-1}\frac{\beta s}{\alpha}\right];$$

so that

$$\int_0^{\frac{\alpha}{\beta}}\beta\sqrt{\alpha^2-\beta^2s^2}\,ds=\frac{\pi}{4}\cdot\frac{\alpha^2}{\beta}.$$

and $$\int_0^1\beta\sqrt{\alpha^2-\beta^2s^2}\,ds=\tfrac{1}{2}\sqrt{\alpha^2-\beta^2}+\tfrac{1}{2}\frac{\alpha^2}{\beta}\sin^{-1}\frac{\beta}{\alpha}$$

Hence

$$V=\tfrac{1}{2}\pi abc\int_{0,\,\theta_0}^{\theta'}\frac{d\theta}{\sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)}}\left(1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right)$$

$$+ab^2cy\int_{\theta'}^{\infty}\frac{d\theta}{(b^2+\theta)\sqrt{\theta(a^2+\theta)(c^2+\theta)}}\Bigg[\Big(1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{b^2y^2}{\theta(b^2+\theta)}\Big)^{\frac{1}{2}}$$

$$+\frac{\Big(1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\Big)\sqrt{\theta(b^2+\theta)}}{by}\sin^{-1}\frac{by}{\sqrt{\theta(b^2+\theta)}\Big(1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\Big)^{\frac{1}{2}}}\Bigg].$$

The lower limit of the first integral being 0 or θ_0 according as the point is inside or outside the ellipsoid, θ_0 and θ' being defined as in §4. It should be remembered that the potential of the semi ellipsoid

P $=\frac{1}{2}$ potential of the complete ellipsoid + V.

6. Now, going back to the integral in §3 let us suppose that α, β, γ do not vanish. We have to deal with the integral

$$\int^{\infty} e^{-\frac{v^2}{4}\frac{b^2\theta}{(b^2+\theta)(q+iw)}}\sin v\left\{\frac{b^2y}{b^2+\theta}+\frac{1}{2}\frac{\beta b^2\theta}{(b^2+\theta)(q+iw)}\right\}\frac{dv}{v}.$$

This by the substitution

$$\left\{\frac{b^2y}{b^2+\theta}+\frac{1}{2}\frac{\beta b^2\theta}{(b^2+\theta)(q+iw)}\right\}v=w$$

transforms into

$$\sqrt{\pi}\,\frac{b\,y\,(q+iw)+\frac{1}{2}\beta b^2\theta}{b\sqrt{\theta(b^2+\theta)}\sqrt{q+iw}}\int^1 e^{-\frac{[b^2y(q+iw)+\frac{1}{2}\beta b^2\theta]^2}{b^2(b^2+\theta)\theta(q+iw)}s^2}\,ds.$$

Hence

$$V=\frac{\Gamma(\lambda)}{2m\Gamma\left(\frac{m}{2}\right)}abc\int_0\int_0\frac{e^{\frac{1}{2}-\frac{m}{2}}}{(b^2+\theta)\sqrt{\theta(a^2+\theta)(c^2+\theta)}}$$

$$\alpha\frac{a^2x}{a^2+\theta}+\beta\frac{b^2y\,(1-s^2)}{b^2+\theta}+\gamma\frac{c^2z}{c^2+\theta}\,Q\,d\theta ds\,F.$$

where

$$I=\int^{\infty}[b^2y(q+iw)+\tfrac{1}{2}\beta b\theta]\frac{e^{(q+iw)\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{b^2s^2y^2}{\theta(b^2+\theta)}\right]}}{(q+iw)^{\lambda-\frac{m}{2}+2}}$$

$$\times\, e^{\frac{1}{4(q+wi)}\left[\frac{a^2\theta}{a^2+\theta}\alpha^2+\frac{b^2\theta}{c^2+\theta}(1-s^2)\beta^2+\frac{c^2\theta}{c^2+\theta}\gamma^2\right]}\,dw$$

$$=\int^{\infty}\left[\frac{by}{(q+iw)^{\lambda-\frac{m}{2}+1}}+\frac{1}{2}\frac{\beta b\theta}{(q+iw)^{\lambda-\frac{m}{2}+2}}\right]$$

$$e^{(q+iw)\left[1-\frac{x^2}{a^2+\theta}-\text{etc...}\right]}\left[1+\sum_{1}^{\infty}\frac{1}{n!}\frac{A^n}{(q+iw)^n}\right]dw.$$

where

$$A=\frac{\theta}{4}\left[\frac{a^2}{a^2+\theta}\alpha^2+\frac{b^2\beta^2}{b^2+\theta}(1-s^2)+\frac{c^2\gamma^2}{c^2+\theta}\right]=Q_1+Q_2,$$

where $Q_1=by\int e^{(q+iw)\left[1-\frac{x^2}{a^2+\theta}-\text{etc.}\right]}\left\{\frac{1}{(q+iw)^{\lambda-\frac{m}{2}+1}}\right.$

$$\left.+\sum\frac{A^n}{\underline{|n}\,(q+iw)^{\lambda-\frac{m}{2}+n+1}}\right\}dw$$

$$=by\frac{2\pi}{\Gamma(\lambda-\frac{m}{2}+1)}R^{\lambda-\frac{m}{2}}\left[1-\frac{RB\theta}{2(2\lambda-m+2)}\right.$$

$$\left.+\frac{R^2B^2\theta^2}{2\cdot4\cdot(2\lambda-m+2)(2\lambda-m+4)}+\dots\right]\dots\quad(\lambda-\tfrac{m}{2}+1>0)$$

where R is as defined in §3 and

$$B=\frac{4A}{\theta}=\frac{a^2}{a^2+\theta}\alpha^2+\frac{b^2\beta^2}{b^2+\theta}(1-s^2)+\frac{c^2\gamma^2}{c^2+\theta}\quad\text{and also}$$

when R is positive, for negative values of R the integral vanishes, and in a similar manner

$$Q_2=\tfrac{1}{2}\beta\, b\theta\ \frac{2\pi}{\Gamma(\lambda-\frac{m}{2}+2)}\ R^{\lambda-\frac{m}{2}+1}\left[1+\frac{RB\theta}{2.\,(2\lambda-m+4)}+\ldots\right]\ldots$$

where R is positive, and is zero when R is negative.

Now replacing α, β, γ by $\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}$ and allowing the whole integral to operate on $F\left(\frac{x'}{a}, \frac{y'}{b}, \frac{z'}{c}\right)$ and finally putting x', y', z' equal to zero after differentiation we have

$$V=V_1+V_2$$

where

$$V_1=\frac{1}{m}\frac{\Gamma(\lambda)\,\pi ab^2c}{\Gamma(\frac{m}{2})\Gamma(\lambda-\frac{m}{2}+1)}\,y\iint\frac{\theta^{\frac{m}{2}-}}{(b^2+\theta)\sqrt{\theta(a^2+\theta)(c^2+\theta)}}$$

$$\times R^{\lambda-\frac{m}{2}}\left[1+\frac{RL\theta}{2.\,(2\lambda-m+4)}+\ldots\right]$$

$$F\left(\frac{ax}{a^2+\theta},\ \frac{by}{b^2+\theta}(1-s^2),\ \frac{cz}{c^2+\theta}\right)ds\,d\theta$$

and
$$V_2=\frac{1}{2m}\frac{\Gamma(\lambda)}{\Gamma(\frac{m}{2})}\cdot\frac{\pi ac}{\Gamma(\lambda-\frac{m}{2}+2)}\iint\frac{\theta^{1-\frac{m}{2}}}{(1-s^2)\sqrt{(a^2+\theta)(c^2+\theta)}}$$

$$\times R^{\lambda-\frac{m}{2}+1}\left[1+\frac{RL\theta}{2.\,(2\lambda-m+4)}+\ldots\right]$$

$$\frac{\partial}{\partial y}F\left(\frac{ax}{a^2+\theta},\ \frac{by}{b^2+\theta}(1-s^2),\ \frac{cz}{c^2+\theta}\right)ds\,d\theta$$

where

$$L=\frac{a^2+\theta}{a^2}\frac{\partial^2}{\partial x^2}+\frac{b^2+\theta}{b^2(1-s^2)}\frac{\partial^2}{\partial y^2}+\frac{c^2+\theta}{c^2}\frac{\partial^2}{\partial z^2}.$$

and as before the integrations are to be extended over those values of θ and s which make

$$R=1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}>0\ (1\geq s\geq 0,\ \theta \text{ positive}).$$

7. The case of the semi-elliptic plate comes in this connection and can be treated exactly in the same way. By using the same discontinuous factors and after a similar series of processes the following results may be obtained.

For a semi-elliptic plate of density σ where

$$\sigma=\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}\right)^{\lambda-1} \text{ on the half } \eta>0,$$

the potential $P=\frac{1}{2}P_1+V$, where P_1 is the potential of the entire elliptic plate of this density and

$$=\frac{1}{m}\frac{\Gamma(\lambda)\quad\sqrt{\pi a b^2}}{\Gamma(\frac{m}{2})\Gamma(\lambda-\frac{m}{2}+\frac{1}{2})}\,y\iint\frac{\theta^{-\frac{m}{2}-\frac{1}{2}}}{(\theta+b^2)(\theta+a^2)^{\frac{1}{2}}}$$

$$\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}\right]^{\lambda-\frac{m}{2}-\frac{1}{2}}d\theta ds$$

where the integrations are to be taken for all values of s and θ which make

$$1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{\theta}-\frac{s^2b^2y^2}{\theta(b^2+\theta)}>0. \quad [1\geq s\geq 0,\ \theta \text{ positive}].$$

The discussion about the limits of integration given in §4 would also apply in this case. Supposing the plate to be homogeneous, the potential under Newtonian law of force is given by

$$V=ab^2y\iint\frac{d\theta ds}{\theta(b^2+\theta)(a^2+\theta)}, \text{ the limits being properly taken}$$

$$=ab\int_{0,\theta_0}^{\theta'}\frac{d\theta}{\sqrt{\theta(a^2+\theta)(b^2+\theta)}}\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{\theta}\right]^{\frac{1}{2}}$$

$$+ab^2y\int_{\theta'}^{\infty}\frac{d\theta}{\theta(b^2+\theta)(a^2+\theta)^{\frac{1}{2}}}.$$

according to the scheme of integration given in § 4, and the lower limit of the first integration being 0 or θ_0 according as the point is inside or outside the plate and θ_0 and θ' are the greatest (positive) roots of the equations

$$1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{\theta}=0$$

and

$$1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}\left(1+\frac{b^2}{\theta}\right)-\frac{z^2}{\theta}=0$$

respectively.

The case of the density

$$\sigma=\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}\right)^{\lambda-1} F\left(\frac{\xi}{a},\frac{\eta}{b}\right)$$

can be treated exactly as the corresponding three dimensional problem and results analogous to those given in §6 can be obtained.

8. Having done with the semi-ellipsoid we propose to take up the case of an incomplete ellipsoid bounded by a plane parallel to one of the principal planes. By slightly altering the second discontinuous factor given in §2 the potential of the incomplete figure may be obtained by an exactly similar analysis. Supposing that the solid under contemplation is greater than a semi-ellipsoid and is limited on the upper part by the plane $\eta=\kappa$ the suitable discontinuous factor would be

$$\frac{1}{\pi}\int_0^\infty \frac{\sin(\kappa-\eta)r}{r}\,dr$$

which is $\frac{1}{2}$ or $-\frac{1}{2}$ according as $\eta<\kappa$ or $>\kappa$.

If the density is given by the function

$$\sigma=\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)^{\lambda-1} F\left(\frac{\xi}{a},\frac{\eta}{b},\frac{\zeta}{c}\right)$$

the potential P of the figure is given by

$$P=\tfrac{1}{2}P_1+V$$

where P_1 is the potential of the complete ellipsoid of the same density and

$$V=\frac{1}{m}\,\frac{\Gamma(\lambda)}{2\pi^2}\iiiint\!\!\int \sin(\kappa-\eta)r\,.\,\frac{e^{\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)(q+iw)}}{(q+iw)}$$

$$\frac{e^{\alpha\xi+\beta\eta+\gamma\zeta}}{r^m}\,d\xi d\eta d\zeta dr dw\;F.$$

and it may also be shown as in § 6 that

$$V=V_1+V_2.$$

where

$$V_1 = \frac{1}{m} \frac{\Gamma(\lambda)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\lambda-\frac{m}{2}+1\right)} \pi ac \iint \frac{[k(b^2+\theta)-b^2y]\theta^{-\frac{m}{2}}}{(b^2+\theta)\sqrt{(a^2+\theta)(c^2+\theta)}}$$

$$\times R'^{\lambda-\frac{m}{2}}\left[1+\frac{R'L'\theta}{2\cdot(2\lambda-m+2)}+\ldots\ldots\ldots\right].$$

$$F\left(\frac{ax}{a^2+\theta}, \frac{by}{b^2+\theta}(1-s^2)+\frac{ks^2}{b}, \frac{cz}{c^2+\theta}\right) ds\, d\theta$$

and

$$V_2 = -\frac{1}{2m} \frac{\Gamma(\lambda)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\lambda-\frac{m}{2}+2\right)} \pi ac \iint \frac{\theta^{1-\frac{m}{2}}}{(1-s^2)\sqrt{(a^2+\theta)(c^2+\theta)}}$$

$$\times R'^{\lambda-\frac{m}{2}+1}\left[1+\frac{R'L'\theta}{2(2\lambda-m+4)}+\ldots\ldots\ldots\right]$$

$$\frac{\partial}{\partial y} F\left(\frac{ax}{a^2+\theta}, \frac{by}{b^2-\theta}(1-s^2)+\frac{ks^2}{b}, \frac{cz}{c^2+\theta}\right) ds\, d\theta$$

where

$$L' = \frac{a^2+\theta}{a^2}\frac{\partial^2}{\partial x^2} + \frac{b^2+\theta}{b^2(1-s^2)}\frac{\partial^2}{\partial y^2} + \frac{c^2+\theta}{c^2}\frac{\partial^2}{\partial z^2}$$

and the limits of integration are to be determined from

$$R' \equiv 1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{[k(b^2+\theta)-b^2y]^2}{b^2\theta(b^2+\theta)}s^2 > 0,$$

$$\left(\begin{array}{c} 1 \geq s \geq 0 \\ \theta \text{ positive} \end{array}\right).$$

A scheme of integration analogous to that given in §4 may be worked out in this case also. The two dimensional problem may be dealt with in a similar manner.

The potential of a complete homogeneous ellipsoid may be deduced from the results given above. This will also serve as a test of the

accuracy of our results. Putting $\lambda=1$, $m=1$ and $F=1$ the potential under Newtonian law of attraction.

$$P = \tfrac{1}{2} P_1 + 2ac \iint \frac{[k(b^2+\theta)-b^2y]}{(b^2+\theta)\sqrt{\theta(a^2+\theta)(c^2+\theta)}}$$

$$\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{\{k(b^2+\theta)-b^2y\}^2}{b^2\theta(b^2+\theta)}s^2\right]^{\frac{1}{2}} ds\, d\theta$$

[for the limits defined by $R' > 0$, and when the ellipsoid is complete $k=b$. Following the scheme of integration given in §4 with only a slight modification necessary on this occasion it is evident that in the present case θ' will tend to infinity as k tends to b. Hence the integral is equal to

$$\frac{b\sqrt{\theta\, b^2+\theta)}}{b(b^2\theta)-b^2y}\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right]^{\frac{1}{2}}$$

$$2abc\int_{0,\,\theta_0}^{\infty}\int_0 \frac{[b(b^2+\theta)-b^2y]}{b\sqrt{\theta(b^2+\theta)}\,\sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)}}$$

$$\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}-\frac{\{b(b^2+\theta)-b^2y\}^2}{b^2\theta\,(b^2+\theta)}s^2\right]^{\frac{1}{2}} ds\, d\theta.$$

the lower limit of θ being 0 or θ_0 according as the point is inside or outside the ellipsoid while θ_0 is defined in the same way as in §4.

Now since

$$\int \sqrt{a^2-\beta^2s^2}\, ds = \frac{\pi}{4}\frac{a^2}{\beta},$$ the integral will ultimately reduce to

$$\tfrac{1}{2}\pi\, abc\int_{0,\,\theta_0}^{\infty}\frac{d\theta}{\sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)}}\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right]$$

$$=\tfrac{1}{2}P_1.$$

This is as we should expect. In the case of the complete homogeneous elliptic plate under Newtonian law of force the corresponding integral is

$$a\iint \frac{[b(b^2+\theta)-b^2y]}{\theta(b^2+\theta)\sqrt{a^2+\theta}}\, ds\, d\theta$$

$$= ab\iint \frac{[b(b^2+\theta)-b^2y]}{b\sqrt{\theta(b^2+\theta)}} \cdot \frac{1}{\sqrt{\theta(a^2+\theta)(b^2+\theta)}}\, ds\, d\theta$$

while the limits of integration are defined by

$$1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{\theta}-\frac{\{b(b^2+\theta)-b^2y\}^2}{b^2\theta(b^2+\theta)}>0$$

and this by the previous method reduces to

$$ab\int_{0,\,\theta_0}^{\infty} \frac{d\theta}{\sqrt{\theta(a^2+\theta)(b^2+\theta)}}\left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{\theta}\right]$$

which is half the potential of the complete elliptic plate.

9. We now propose to show how the present artifice of evaluating an integral by the use of discontinuous factors may be successfully employed in determining the potential of any part of an ellipsoid and of an elliptic plate. As the procedure is the same in both cases we here deal with only the elliptic plate by way of illustration as it involves simpler calculations than the three dimensional problem which however, does not present any fresh difficulty. To avoid needless complications we assume the density function to be given by

$$=\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}\right)^{\lambda-1}$$

remembering that the case of the more general distribution is to be treated in the usual way. Take the integral

$$\frac{1}{2\pi}\int^{\infty} \frac{\sin(\eta+k\xi+c)v}{v}\, dv+\tfrac{1}{2}$$

It is known that the value of this integral is unity or zero according as the expression $\eta+k\xi+c$ is positive or negative or in other words

according as the point (ξ, η) lies on one side or the other of the straight line

$$\eta+k\xi+c=0.$$

Hence the integral

$$P=\frac{1}{m}\frac{\Gamma(\lambda)}{2\pi}\int^{\infty}\int^{\infty}\frac{d\xi\, d\eta}{r^m}\int^{\infty} e^{\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}\right)(q+iv)}\,\frac{dv}{(q+iv)^{\lambda}}$$

$$\left[\frac{1}{2\pi}\int^{\infty}\frac{\sin(\eta+k\xi+c)v}{v}\,dv+\tfrac{1}{2}\right]$$

gives the potential of that part of an elliptic plate (a, b) of density σ which is bounded by the elliptic arc and the line $\eta+k\xi+c=0$ and encloses the portion in which the expression $\eta+k\xi+c$ is positive. This may be written as

$$P=\tfrac{1}{2}P_1+V$$

where P_1 is the potential of the complete elliptic plate of the same surface density and

$$V=\frac{1}{m}\frac{\Gamma(\lambda)}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\xi\, d\eta}{r^m}\int_{-\infty}^{\infty} e^{\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}\right)(q+iv)}\,\frac{1}{(q+iv)^{\lambda}}$$

$$\int^{\infty}\frac{\sin \eta v \cos (k\xi+c)v}{v}\,dv$$

$$\frac{1}{m}\frac{\Gamma(\lambda)}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\xi\, d\eta}{r^m}\int_{-\infty}^{\infty} e^{\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}\right)(q+iv)}\,\frac{dv}{(q+iv)^{\lambda}}$$

$$\int_{-\infty}^{\infty}\frac{\cos \eta v \sin (k\xi+c)v}{v}\,dv$$

$$=V_1+V_2.$$

Now making use of the integrals

$$\int_{-\infty}^{\infty} e^{-a^2x^2+2bx} \begin{matrix}\sin\\ \cos\end{matrix}\, vx\, dx = \frac{\sqrt{\pi}}{a}\, e^{\frac{b^2}{a^2}-\frac{v^2}{4a^2}} \begin{matrix}\sin\\ \cos\end{matrix}\, \frac{bv}{a^2}$$

it may be shown that $$V_1 = W_1 + W_1',$$

where

$$\begin{bmatrix} W_1 \\ W_1' \end{bmatrix} = \frac{1}{2m} \frac{\Gamma(\lambda)\sqrt{\pi}\, ab}{2m\,\Gamma\left(\frac{m}{2}\right)\Gamma\left(\lambda-\frac{m}{2}+1\right)} \iint \frac{\theta^{-\frac{m}{2}}}{\sqrt{(a^2+\theta)(b^2+\theta)}}$$

$$\frac{\frac{a^2x}{a^2+\theta} \pm \frac{b^2y}{b^2+\theta} \pm c}{\sqrt{\frac{k^2a^2\theta}{a^2+\theta}+\frac{b^2\theta}{b^2+\theta}}} \times \left[1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{\theta}\right.$$

$$\left. - \frac{\left(\frac{a^2x}{a^2+\theta} \pm \frac{b^2y}{b^2+\theta} \pm c\right)^2}{\frac{k^2a^2\theta}{a^2+\theta}+\frac{b^2\theta}{b^2+\theta}}\, \zeta^2\right]^{\lambda-\frac{m}{2}-\frac{1}{2}} d\theta\, ds;$$

wherever the double sign occurs in the integrand the upper sign is to be taken with W_1 and the lower sign with W_1'.

And similarly

$$V_2 = W_1 - W_1'.$$

Hence $$V = 2W_1,$$

where the limits of integration are defined by the relation

$$1-\frac{x^2}{a^2+\theta}-\frac{y^2}{b^2+\theta}-\frac{z^2}{\theta}-\frac{\left(\frac{a^2x}{a^2+\theta}+\frac{b^2y}{b^2+\theta}+c\right)^2}{\frac{k^2a^2\theta}{a^2+\theta}+\frac{b^2\theta}{b^2+\theta}}\, s^2 > 0. \quad \left(\begin{matrix} 1 \geq s \geq 0 \\ \theta \text{ positive}\end{matrix}\right)$$

10. Before conclusion it is necessary to recall the criticism by Kronecker [1] on Dirichlet's method of integration. He observes that however elegant Dirichlet's method may appear to be it is not altogether free from serious difficulties. The change of the order of integration and other points noted by Kronecker are not easily justifiable. These objections would stand in all the examples worked out in this paper. But it may be added that though the process of integration is unsound from the point of rigour it gives quite correctly all the known results about the potentals of uniform ellipsoid and of elliptic plates (§8). It may also be shown to give many other results in connection with problems which may be solved directly. By way of illustration we take a very simple example and propose to calculate the volume of the incomplete ellipsoid of §8 by the present method.

By the ordinary method the volume

$$=\tfrac{2}{3}\pi abc+\pi ac\int_{0}^{k}\left(1-\frac{y^2}{b^2}\right)dy$$

$$=\tfrac{2}{3}\pi abc+\pi ack\left(1-\frac{k^2}{3b^2}\right).$$

Now the use of discontinuous factor gives the volume=vol. of semi-ellipsoid$+\iiint d\xi d\eta d\zeta$ between proper limits.

$$\text{The integral}=\tfrac{1}{2}\pi\int_{-\infty}^{\infty}\frac{e^{\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)(q+iw)}}{(q+iw)}\,dw$$

$$\frac{1}{\pi}\int_{0}^{\infty}\frac{\sin(k-\eta)v}{v}\,dv\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\xi d\eta d\zeta$$

$$=\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{e^{(q+iw)}}{(q+iw)}\,\frac{\pi\sqrt{\pi}\,abc}{(q+iw)^{\frac{3}{2}}}\,\frac{e^{-\frac{b^2v^2}{4(q+iw)}}}{v}\sin vk\,dw\,dv.$$

[1] *l.c. ante.*

But

$$\int_0^\infty \frac{e^{-\frac{b^2v^2}{4(q+iw)}}}{v} \sin vk\,dv = \sqrt{\pi}\,\frac{k}{b}\,(q+iw)^{\frac{1}{2}} \int_0^1 e^{-(q+iw)\frac{k^2}{b^2}s^2}\,ds. \quad (\S 3).$$

Hence the integral

$$= \tfrac{1}{2}ack \int_0^1 \int_{-\infty}^{\infty} \frac{e^{(q+iw)\left(1-\frac{k^2s^2}{b^2}\right)}}{(q+iw)^2}\,ds\,dw$$

$$= \pi ack \int_0^1 \left(1-\frac{k^2}{b^2}s^2\right) ds. \text{ since } 1-\frac{k^2s^2}{b^2} > 0,\ k<b,\ 1 \geq s \geq 0$$

$$= \pi ack \left(1-\frac{k^2}{3b^2}\right)$$

which verifies the result obtained by direct calculation.

On the Wave-Equation in Ellipsoidal Co-ordinates

BY

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1. In a previous paper[1] published in the Bulletin, it was shown that if the ellipsoidal co-ordinates ρ, θ, ϕ be defined by

$$x=a\rho \sin\theta \cos\phi,\ y=b\rho \sin\theta \sin\phi,\ z=c\rho \cos\theta,$$

$$\rho^n\, C_n^m(\theta, \phi) \text{ and } \rho^n\, S_n^m(\theta, \phi),$$

where

$$C_n^m(\theta, \phi)=\frac{(n+m)!}{2\pi\, n!\, i^m}\int_{-\pi}^{\pi}(c\cos\theta+ia\sin\theta\cos\phi\cos u$$

$$+ib\sin\theta\sin\phi\sin u)^n \cos mu\,du$$

and

$$S_n^m(\theta, \phi)=\frac{(n+m)!}{2\pi\, n!\, i^m}\int_{-\pi}^{\pi}(c\cos\theta+ia\sin\theta\cos\phi\cos u$$

$$+ib\sin\theta\sin\phi\sin u)^n \sin mu\,du$$

constitute a class of solutions of the Laplace's equation which are extremely suitable for problems involving an ellipsoidal boundary. The familiar method of determining by means of ordinary spherical harmonics the potential of a spherical bowl or a circular disk at any arbitrary point from the known value of the potential on the axis becomes at once available for solving similar problems for an ellipsoidal bowl or an elliptic plate. The potential of an ellipsoidal bowl or an elliptic plate can be easily obtained at any point on the axis and the potential at any other point can be at once expressed in terms of $\rho^n C_n^m(\theta, \phi)$, $\rho^n S_n^m(\theta, \phi)$, $\rho^{-n-1}\mathfrak{C}_n^m(\theta, \phi)$ or $\rho^{-n-1}\mathfrak{S}_n^m(\theta, \phi)$. The motion of an incompressible liquid in an ellipsoidal cup and many other potential problems can be similarly investigated. The detailed discussion

[1] Vol. X, No. 2, September, 1919.

of these problems will be given in a paper shortly to be published. In particular a method was suggested for constructing a set of solutions of the wave equation

$$(\nabla^2+k^2)\,V=0,$$

in the ellipsoidal co-ordinates (ρ, θ, ϕ) in terms of these harmonics. The possibility of the solution being expressed in the form of a product $\psi_n(k\rho)\,C_n{}^m(\theta, \phi)$ was tacitly assumed in the method. It will be noticed in the light of the further results obtained in this paper that this assumption is not rigorously justifiable, but when the assumption has been made, it is possible to obtain an expression for $\psi_n(k\rho)$ which represents the mean value of the function on the ellipsoid ρ and that with this value for $\psi_n(k\rho)$, the quantity $\psi_n(k\rho)\,C_n{}^m(\theta, \phi)$ very nearly approaches a solution. An approximate treatment of the motion of a gas within a rigid ellipsoidal envelope will be given in some detail towards the end of this paper in illustration of the solutions.

2. If we put $x=ax'$, $y=by'$, $z=cz'$, then the wave equation can be written in the form

$$\frac{1}{a^2}\frac{\partial^2 V}{\partial x'^2}+\frac{1}{b^2}\frac{\partial^2 V}{\partial y'^2}+\frac{1}{c^2}\frac{\partial^2 V}{\partial z'^2}+k^2V=0.$$

If it is assumed that this equation has a solution of the form

$$R_n\,U_n\,,$$

where R_n is a function of ρ only and U_n is a solution of the equation

$$\frac{1}{a^2}\frac{\partial^2 U_n}{\partial x'^2}+\frac{1}{b^2}\frac{\partial^2 U_n}{\partial y'^2}+\frac{1}{c^2}\frac{\partial^2 U_n}{\partial z'^2}=0,$$

then we obtain

$$\frac{1}{a^2}\frac{\partial^2(R_nU_n)}{\partial x'^2}+\frac{1}{b^2}\frac{\partial^2(R_nU_n)}{\partial y'^2}+\frac{1}{c^2}\frac{\partial^2(R_nU_n)}{\partial z'^2}+k^2R_nU_n=0,$$

that is,

$$\left[\frac{1}{a^2}\frac{\partial^2 R_n}{\partial x'^2}+\frac{1}{b^2}\frac{\partial^2 R_n}{\partial y'^2}+\frac{1}{c^2}\frac{\partial^2 R_n}{\partial z'^2}\right]+2\left[\frac{1}{a^2}\frac{\partial R_n}{\partial x'}\frac{\partial U_n}{\partial x'}\right.$$

$$\left.\frac{1}{b^2}\frac{\partial R_n}{\partial y'}\frac{\partial U_n}{\partial y'}+\frac{1}{c^2}\frac{\partial R_n}{\partial z'}\frac{\partial U_n}{\partial z'}\right]+k^2R_nU_n=0.$$

Now since R_n is a function of ρ only, we have

$$\frac{\partial R_n}{\partial x'}=\frac{x'}{\rho}\frac{\partial R_n}{\partial \rho},\quad \frac{\partial R_n}{\partial y'}=\frac{y'}{\rho}\frac{\partial R_n}{\partial \rho},\quad \frac{\partial R_n}{\partial z'}=\frac{z'}{\rho}\frac{\partial R_n}{\partial \rho}.$$

Therefore the above equation becomes

$$\left[\frac{1}{a^2}\frac{\partial}{\partial x'}\left(\frac{x'}{\rho}\frac{\partial R_n}{\partial \rho}\right)+\frac{1}{b^2}\frac{\partial}{\partial y'}\left(\frac{y'}{\rho}\frac{\partial R_n}{\partial \rho}\right)+\frac{1}{c^2}\frac{\partial}{\partial z'}\left(\frac{z'}{\rho}\frac{\partial R_n}{\partial \rho}\right)\right]U_n$$

$$+\frac{2}{\rho}\frac{\partial R_n}{\partial \rho}\left[\frac{x'}{a^2}\frac{\partial U_n}{\partial x'}+\frac{y'}{b^2}\frac{\partial U_n}{\partial y'}+\frac{z'}{c^2}\frac{\partial U_n}{\partial z'}\right]+k^2R_nU_n=0.$$

This can also be written in the form

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial R_n}{\partial \rho}\right)\left[\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}\right]U_n$$

$$+\frac{1}{\rho}\frac{\partial R_n}{\partial \rho}\left[2\left(\frac{x'}{a^2}\frac{\partial U_n}{\partial x'}+\frac{y'}{b^2}\frac{\partial U_n}{\partial y'}+\frac{z'}{c^2}\frac{\partial U_n}{\partial z'}\right)\right.$$

$$\left.+U_n\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\right]+k^2R_nU_n=0.$$

It will appear from this equation that it is not possible to separate the differential equation for R_n. But it is easy to obtain the differential equation satisfied by the mean value of R_n on any ellipsoidal surface. If we put $U_n=\rho^n C_n(\theta,\phi)$, this becomes

$$\rho\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial R_n}{\partial \rho}\right)\frac{C_n(\theta,\phi)}{p_n^2}+\frac{1}{\rho}\frac{\partial R_n}{\partial \rho}\left[2n\,D_n(\theta,\phi)\right.$$

$$\left.+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)C_n(\theta,\phi)\right]+k^2R_nC_n(\theta,\phi)=0.$$

where $$\frac{1}{p_n}=\frac{\sin^2\theta\cos^2\phi}{a^2}+\frac{\sin^2\theta\sin^2\phi}{b^2}+\frac{\cos^2\theta}{c^2}$$

and

$$D_n^m(\theta,\phi)=\frac{(n+m)!}{2\pi n!\,i^m}\int_{-\pi}^{\pi}(c\cos\theta+ia\sin\theta\cos\phi\cos u$$

$$+ib\sin\theta\sin\phi\sin u)^{n-1}$$

$$\left(\frac{\cos\theta}{c}+\frac{i\sin\theta\cos\phi\cos u}{a}+\frac{i\sin\theta\sin\phi\sin u}{b}\right)du.$$

Multiplying this equation by the conjugate function $C_n(\theta, \phi)$ and integrating we obtain

$$\rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho} \frac{\partial R_n}{\partial \rho}\right) \int_0^{\pi}\int_0^{2\pi} [C_n(\theta, \phi)]^2 \frac{\sin\theta}{p_n^2}\, d\theta d\phi$$

$$+\frac{1}{\rho} \frac{\partial R_n}{\partial \rho}\left[2n \int_0^{\pi}\int_0^{2\pi} D_n(\theta, \phi)\, C_n(\theta, \phi) \sin\theta d\theta d\phi\right.$$

$$\left.+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\int_0^{\pi}\int_0^{2\pi} [C_n(\theta, \phi)]^2 \sin\theta d\theta d\phi\right]$$

$$+k^2 R_n \int_0^{\pi}\int_0^{2\pi} [C_n(\theta, \phi)]^2 \sin\theta d\theta d\phi = 0.$$

Now it is easy to see that

$$(2n+3)\int_0^{\pi}\int_0^{2\pi} [C_n(\theta, \phi)]^2 \frac{\sin\theta}{p_n^2}\, d\theta d\phi$$

$$=2n\int_0^{\pi}\int_0^{2\pi} D_n(\theta, \phi)\, C_n(\theta, \phi) \sin\theta d\theta d\phi$$

$$+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\int_0^{\pi}\int_0^{2\pi} [C_n(\theta, \phi)]^2 \sin\theta d\theta d\phi.$$

Hence the differential equation for R_n reduces to the form

$$\rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho} \frac{\partial R_n}{\partial \rho}\right) + (2n+3)\left(\frac{1}{\rho} \frac{\partial R_n}{\partial \rho}\right)$$

$$+ k^2 \frac{\displaystyle\int_0^{\pi}\int_0^{2\pi} [C_n(\theta, \phi)]^2 \sin\theta d\theta d\phi}{\displaystyle\int_0^{\pi}\int_0^{2\pi} [C_n(\theta, \phi)]^2 \frac{\sin\theta}{p_n^2}\, d\theta d\phi}\, R_n = 0.$$

If now we write

$$k^2 \frac{\int_0^\pi \int_0^{2\pi} [C_n(\theta, \phi)]^2 \sin\theta d\theta d\phi}{\int_0^\pi \int_0^{2\pi} [C_n(\theta, \phi)]^2 \frac{\sin\theta}{p_n^2} d\theta d\phi} = k'^2,$$

the equation can be written in the form

$$\rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial R_n}{\partial \rho}\right) + (2n+3)\frac{1}{\rho}\frac{\partial R_n}{\partial \rho} + k'^2 R_n = 0,$$

which has the well-known solution

$$R_n = A\psi_n(k'\rho) + B\Psi_n(k'\rho).$$

where A and B are two arbitrary constants and

$$\psi_n(\rho) = \left(-\frac{1}{\rho}\frac{\partial}{\partial \rho}\right)^n \frac{\sin\rho}{\rho}, \quad \Psi_n(\rho) = \left(-\frac{1}{\rho}\frac{\partial}{\partial \rho}\right)^n \frac{\cos\rho}{\rho}.$$

Also

$$\rho^n \psi_n(\rho) = \sqrt{\frac{\pi}{2\rho}}\, J_{n+\frac{1}{2}}(\rho),$$

$$\rho^n \Psi_n(\rho) = \sqrt{\frac{\pi}{2\rho}}\, J_{-n-\frac{1}{2}}(\rho).$$

The condition, which any two distinct solutions V, V′ of the wave equation, which themselves or their differential co-efficients in the direction of the normal vanish on the surface of the ellipsoid, must satisfy, namely

$$\int_0^1 \int_0^\pi \int_0^{2\pi} VV'\rho^2 \sin\theta d\rho d\theta d\phi = 0,$$

is also easily seen to be satisfied by these approximate solutions.

3. One of the most interesting applications of these results is to the investigation of the motion of a gas within a rigid ellipsoidal envelope. To determine the free periods we have only to suppose that $\frac{\partial\psi}{\partial\rho}$ vanishes when $\rho=1$. The symmetrical vibrations in which the disturbance on each similar and similarly situated ellipsoidal surface is in the same phase will be determined by $\psi_o\ (k'\rho)$ which satisfies the equation

$$\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi_o}{\partial\rho}\right)+\frac{3}{\rho}\frac{\partial\psi_o}{\partial\rho}+\frac{3k^2}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}\psi_o=0.$$

Therefore

$$\psi_o=\frac{\sin\left(\frac{\sqrt{3}k\rho}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}\right)}{\rho}$$

The free periods are given by

$$\frac{\sqrt{3}k\rho}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}=1\cdot4303\pi,\ 2\cdot4590\pi\ 3\cdot4709\pi,\ 4\cdot4774\pi,\ 5\cdot4818\pi,\ 6\cdot4844\pi,\ \text{etc.}$$

The first finite root corresponds to the symmetrical vibration of lowest pitch. In the case of a higher root the vibrations in question has ellipsoidal nodes defined by the values of ρ corresponding to the inferior roots. It will be noticed that the pitch would be lower for the ellipsoidal shell than for a corresponding spherical shell obtained by putting $a=b=c=1$. The amount by which the pitch is decreased for an ellipsoidal shell of given dimensions can be easily calculated from the above formula.

4. The case of $n=1$ is perhaps the most interesting. The differential equation satisfied by ψ_1 is

$$\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi_1}{\partial\rho}\right)+\frac{5}{\rho}\frac{\partial\psi_1}{\partial\rho}+k'^2\psi_1=0,$$

where

$$k'^2=\frac{5k^2}{\frac{3}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}},\quad\frac{5k^2}{\frac{1}{a^2}+\frac{3}{b^2}+\frac{1}{c^2}}\quad\text{or}\quad\frac{5k^2}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{3}{c^2}}.$$

Hence the vibration at any point is given by

$$-\frac{\partial}{\partial(k'\rho)}\frac{\sin k'\rho}{k'\rho}\mathrm{U}_1,$$

where $\mathrm{U}_1=a\sin\theta\cos\phi$, $b\sin\theta\sin\phi$, or $c\cos\theta$. Hence the air sways from side to side in the directions of the three principal axes. For vibrations in the direction of the a—axis, the periods are given by

$$\frac{\sqrt{5}k}{\sqrt{\left(\frac{3}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}=\cdot 6625\pi,\ 1\cdot 8908\pi,\text{ etc.},$$

for vibrations in the direction of the b-axis by

$$\frac{\sqrt{5}k}{\sqrt{\left(\frac{1}{a^2}+\frac{3}{b^2}+\frac{1}{c^2}\right)}}=\cdot 6625\pi,\ 1\cdot 8908\pi,\text{ etc.},$$

and for the direction of the c-axis by

$$\frac{\sqrt{5}k}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{3}{c^2}\right)}}=\cdot 6625\pi,\ 1\cdot 8908\pi,\text{ etc.}$$

5. When $n=2$, the differential equation satisfied by ψ_2 is

$$\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\psi_2}{\partial\rho}\right)+\frac{7}{\rho}\frac{\partial\psi_2}{\partial\rho}+k'^2\psi_2=0,$$

where $k'^2=$

$$\frac{7k^2\left[12a^4+3(b^4+c^4)-4a^2(b^2+c^2)+2b^2c^2\right]}{8(b^2+c^2+4a^2)+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left[12a^4+3(b^4+c^4)-4a^2(b^2+c^2)+2b^2c^2\right]}$$

$$\frac{7k^2\left[12b^4+3(c^4+a^4)-4b^2(c^2+a^2)+2c^2a^2\right]}{8(c^2+a^2+4b^2)+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left[12b^4+3(c^4+a^4)-4b^2(c^2+a^2)+2c^2a^2\right]}$$

or

$$\frac{7k^2\left[12c^4+3(a^4+b^4)-4c^2(a^2+b^2)+2a^2b^2\right]}{(a^2+b^2+4c^2)+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left[12c^4+3(a^4+b^4)-4c^2(a^2+b^2)+2a^2b^2\right]}$$

The spherical nodes are given by

$$\tan k'\rho=\frac{k'^3\rho^3-2k'\rho}{4k'^2\rho^2-9}$$

of which the first finite solution is $k'\rho=3\cdot3422$, giving a tone graver than any of the symmetrical group. The following will also be seen to be nodal surfaces

$$2x^2-y^2-z^2=0,\ 2y^2-z^2-x^2=0,\ 2z^2-x^2-y^2=0.$$

It will appear from the above results that corresponding to a single mode of vibration of the gas inside a spherical shell we get three distinct modes of vibration for the ellipsoidal shell. This result is also clear from the general expression. The periods for the nth mode are determined by k which are the roots of the equation

$$\frac{\partial}{\partial\rho}\left[\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\frac{\sin k'\rho}{\rho}\right]=0,$$

where

$$k'^2=k^2\frac{\int_0^\pi\int_0^{2\pi}[C_n(\theta,\phi)]^2\sin\theta\,d\theta\,d\phi}{\int_0^\pi\int_0^{2\pi}[C_n(\theta,\phi)]^2\frac{\sin\theta}{p_n^2}d\theta\,d\phi}$$

It is clear from the above expression that by an interchange of the letters a, b, c in the expression for $C_n(\theta,\phi)$ we get three distinct types of vibrations.

6. As we are not yet in possession of any rigorous solution of the wave-equation either in the confocal system λ,μ,ν or in the system ρ,θ,ϕ, it is thought that the approximate solutions given above may be used with advantage to elucidate some of the obscure points in the ellipsoidal problem. One advantage of the solution is that the methods already in common use for the spherical problems can be easily extended to solve analogous problems for the ellipsoidal boundary.

Further results on the harmonics $C_n^m(\theta,\phi)$, $S_n^m(\theta,\phi)$ and the solutions of the wave-equation and their applications to other physical problems will be published in due course.

On the numerical calculation of the roots of the equations $P_n^m(\mu)=0$ and $\frac{d}{d\mu}P_n^m(\mu)=0$ regarded as equations in n.

[Part II]

BY

Bholanath Pal

In my first paper,[1] "On the numerical calculation of the roots of the equations $P_n^m(\mu)=0$ and $\frac{d}{d\mu}P_n^m(\mu)=0$ regarded as equations in n," I promised to give some further tables for the values of n, for, $\theta=\frac{\pi}{4}$, $\theta=\frac{\pi}{6}$ and $\theta=\frac{\pi}{12}$. These tables have now been completed and are given in the following pages.

The method which I followed in calculating the roots was, as I explained in my first paper, derived from an asymptotic expansion of $P_n^m(\mu)$ as a function of n recently given by Prof. Watson in the "Transactions of the Cambridge Philosophical Society" (October, 1918). I should also point out that the expressions for n for which these functions vanish and which I have obtained from the asymptotic expansion are rapidly convergent and very convenient for numerical work.

The values of n for which $P_n^m(\mu)$ vanishes are given by

$$n=\xi+\frac{1}{\theta}\left\{a_1+a_2+a_3-\frac{a_1^3}{3}-a_1^2a_2-\dots\right\}$$

$$+\frac{1}{\theta^2}\left\{a_1a'_1+a'_1a_2+a_1a'_2+\dots\right\}$$

[1] Bulletin of the Calcutta Mathematical Society, Vol. IX, No. 2, March (1919).

where

$$\xi=\frac{\pi}{2\theta}\left\{2k-m+\frac{3}{2}-\frac{\pi}{\theta}\right\},$$

$$a_1=-\frac{C'_1}{2\xi},$$

$$a_2=\frac{C'_1C_1-3C'_2}{(2\xi)^2}.$$

$$a_3=\frac{3C_2C'_1+3C'_2C_1-C_1{}^2C'_1-15C'_3}{(2\xi)^3}.$$

... ...

... ...

and

$$C_1=m^2-\tfrac{1}{4},$$

$$C'_1=(m^2-\tfrac{1}{4})\cot\theta,$$

$$C_2=\tfrac{1}{8}(m^2-\tfrac{1}{4})^2-\tfrac{1}{6}(m^2-\tfrac{9}{4})(m^2-\tfrac{1}{4})\cot^2\theta,$$

$$C'_2=\tfrac{1}{3}(m^2-\tfrac{9}{4})(m^2-\tfrac{1}{4})\cot\theta,$$

... ...

... ...

and $a'_1, a'_2\ldots$ are written for $\dfrac{da_1}{d\xi}, \dfrac{da_2}{d\xi}, \ldots$

The roots of the equation $P_n{}^m(\mu)=0$.

TABLE I

$$\theta=\frac{\pi}{4},\ m=0$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	6·5	·0192	−·0014	−·0000	−·0000	·0000	·0000	6·52
2	10·5	·0119	−·0005	−·0000	−·0000	·0000	·0000	10·51
3	14·5	·0086	−·0002	−·0000	−·0000	·0000	·0000	14·51
4	18·5	·0067	−·0001	−·0000	−·0000	·0000	·0000	18·50
5	22·5	·0055	−·0001	−·0000	−·0000	·0000	·0000	22·50

The roots of the equation $P_n^m(\mu)=0$

TABLE II

$$\theta=\frac{\pi}{4}, \; m=1$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	4·5	—·0833	·0092	·0000	—·0015	·0003	·0001	4·40
2	8·5	—·0441	·0025	·0000	—·0002	·0000	·0000	8·44
3	12·5	—·0300	·0012	·0000	—·0000	·0000	·0000	12·46
4	16·5	—·0225	·0007	·0000	—·0000	·0000	·0000	16·47
5	20·5	—·0182	·0004	·0000	—·0000	·0000	·0000	20·47

TABLE III

$$\theta=\frac{\pi}{4}, \; m=2$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	2·5	—·7500	·1500	—·0190	—·2250	0910	·0450	1·52
2	6·5	—·2884	·0221	—·0011	—·0138	·0022	·0011	6·15
3	10·5	—·1785	·0085	—·0002	—·0033	·0002	·0001	10·28
4	14·5	—·1293	·0039	—·0000	—·0010	·0000	·0000	14·34
5	18·5	—·1013	·0029	—·0000	—·0005	·0000	·0000	18·37

TABLE IV

$$\theta=\frac{\pi}{6}, \; m=0$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	10	·0216	—·0010	—·0000	—·0000	0000	·0000	10·03
2	16	·0135	—·0004	—·0000	—·0000	0000	·0000	16·02
3	22	·0098	—·0002	—·0000	—·0000	0000	·0000	22·01
4	28	·0077	—·0001	—·0000	—·0000	0000	·0000	28·01
5	34	·0063	—·0000	—·0000	—·0000	0000	·0000	34·01

The roots of the equation $P_n^m(\mu)=0$.

TABLE V

$$\theta=\frac{\pi}{6}, \quad \iota=1$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	7	—·0927	·0066	·0000	—·0013	·0001	·0000	6·83
2	13	—·0499	·0019	·0000	—·0001	·0000	·0000	12·91
3	19	—·0341	·0009	·0000	—·0000	·0000	·0000	18·93
4	25	—·0259	·0005	·0000	—·0000	·0000	·0000	24·95
5	31	—·0209	·0003	·0000	—·0000	·0000	·0000	30·96

TABLE VI

$$\theta=\frac{\pi}{6}, \quad m=2$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	4	—·8118	·1014	—·1231	—·1652	·0143	·0206	2·26
2	10	—·3247	·0162	—·0082	—·0105	·0011	·0005	9·39
3	16	—·2027	·0063	—.0031	—·0025	·0001	·0000	15·62
4	22	—·1476	·0033	—·0012	—·0009	·0000	·0000	21·72
5	28	—·1159	·0020	—·0007	—·0004	·0000	·0000	27·78

TABLE VII

$$\theta=\frac{\pi}{12}, \quad m=0$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	20·5	·0227	—·0065	—·0000	—·0000	·0000	·0000	20·60
2	32·5	·0143	—·0002	—·0000	—·0000	·0000	.0000	32·55
3	44·5	·0104	—·0001	—·0000	—·0000	·0000	·0000	44·53
4	56·5	·0082	—·0000	—·0000	—·0000	·0000	·0000	56·53
5	68·5	·0068	—·0000	—·0000	—·0000	·0000	·0000	68·52

The roots of the equation $P_n^m(\mu)=0$.

TABLE VIII

$$\theta=\frac{\pi}{12}\,.\; m=1$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	14·5	−·0965	·0033	·0003	−·0006	·0090	·0000	14·14
2	26·5	−·0528	·0009	·0000	−·0000	·0000	·0000	26·30
3	38·5	−·0363	·0004	·0000	−·0000	·0000	·0000	38·36
4	50·5	−·0277	·0002	·0000	−·0000	·0000	·0000	50·39
5	62·5	−·0223	·0001	·0000	−·0000	·0000	·0000	62·41

TABLE IX

$$\theta=\frac{\pi}{12}\,.\; m=2$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	8·5	−·8221	·0484	−·0723	−·0877	·0103	·0051	4·77
2	20·5	−·3414	·0083	−·0061	−·0057	·0003	·0001	19·21
3	32·5	−·2153	·0033	−·0023	−·0015	·0000	·0000	31·67
4	44·5	−·1572	·0017	−·0006	−·0005	·0000	·0000	43·90
5	56·5	−·1238	·0010	−·0000	−·0000	·0000	·0000	56·03

The values of n which make $\frac{d}{d\mu}P_n^m(\mu)$ vanish are given by

$$n=\xi+\frac{1}{\theta}\left\{a_1+a_2+a_3-\frac{a_1^3}{3}-a_1^2a_2-\ldots\right\}+\frac{1}{\theta^2}\{a_1a'_1+a'_1a_2$$

$$\ldots \qquad\qquad \ldots \qquad +a_1a'_2+\ldots\}$$

where

$$\xi=\frac{\pi}{2\theta}\left(2k-m+\frac{1}{2}-\frac{\theta}{\pi}\right),$$

$$a_1=\frac{a_0}{a'_0}$$

$$a_2=\left(\frac{a_1}{a'_0}-\frac{a_0a'_1}{a'^2_0}\right)\frac{1}{2\xi},$$

$$a_3=\left(\frac{3a_2}{a'_0}-\frac{a_1a'_1}{a'^2_0}-\frac{3a'_2a_0}{a'^2_0}+\frac{a'^2_1a_0}{a'^3_0}\right)\frac{1}{(2\xi)^2},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

and

$$a_0=-\tfrac{1}{2}\cot\theta,$$

$$a'_0=(\xi+\tfrac{1}{2}),$$

$$a_1=-(\xi+\tfrac{1}{2})(m^2-\tfrac{1}{4})\cot\theta-\tfrac{1}{2}(m^2-\tfrac{1}{2})\cot\theta,$$

$$a'_1=(\xi+\tfrac{1}{2})(m^2-\tfrac{1}{2})-(m^2-\tfrac{1}{4})-\tfrac{3}{2}(m^2-\tfrac{1}{4})\cot^2\theta,$$

$$a_2=-\tfrac{1}{3}(\xi+\tfrac{1}{2})(m^2-\tfrac{3}{2})(m^2-\tfrac{1}{4})\cot\theta+\tfrac{1}{3}(m^2-\tfrac{9}{4})(m^2-\tfrac{1}{4})\cot\theta$$

$$-\tfrac{1}{12}(m^2-\tfrac{1}{2})^2\cot\theta+\tfrac{5}{12}(m^2-\tfrac{9}{4})(m^2-\tfrac{1}{4})\cot^3\theta,$$

$$a'_2=\tfrac{1}{6}(\xi+\tfrac{1}{2})(m^2-\tfrac{1}{2})^2-\tfrac{1}{3}(m^2-\tfrac{3}{2})(m\;-\tfrac{1}{4})$$

$$-\tfrac{1}{6}(\xi+\tfrac{1}{2})(m^2-\tfrac{1}{4})(m^2-\tfrac{9}{4})\cot^2\theta-\tfrac{1}{12}(m^2-\tfrac{3}{2})(m^2-\tfrac{1}{4})\cot^2\theta,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

and $a'_1, a'_2, \ldots$ are written for $\dfrac{da_1}{d\xi}, \dfrac{da_2}{d\xi} \;\cdots\;\cdots$

The roots of the equation $\frac{d}{d\mu} P_n^m(\mu)=0$

TABLE X

$\theta=\frac{\pi}{6}$, $m=0$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	7	—·1154	·0309	—·0003	—·0018	·0006	·0005	6·83
2	13	—·0641	·0166	—·0002	—·0002	·0001	·0001	12·89
3	19	—·0444	·0113	—·0000	—·0001	·0000	0000	18·93
4	25	—·0339	·0086	—·0000	—·0000	·0000	·0000	24·95
5	31	—·0274	·0069	—·0000	—·0000	·0000	·0000	30·96

TABLE XI

$\theta=\frac{\pi}{6}$, $m=-1$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	10	—·0824	—·0665	·0014	—·0006	—·0005	—·0004	9·71
2	16	—·0524	—·0410	·0005	—·0002	—·0001	—·0001	15·82
3	22	—·0385	—·0296	·0003	—·0000	—·0000	—·0000	21·87
4	28	—·0303	—·0232	·0001	—·0000	—·0000	—·0000	27·89
5	34	—·0251	—·0191	·0000	—·0000	—·0000	—·0000	33·91

TABLE XII

$\theta=\frac{\pi}{6}$, $m=-2$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	13	—·0641	—·2535	—·0000	—·0002	—·0015	—·0013	12·38
2	19	—·0444	—·1733	·0015	—·0001	—·0005	—·0004	18·58
3	25	—·0339	—·1303	·0012	—·0000	—·0003	—·0002	24·68
4	31	—·0274	—·1047	·0008	—·0000	—·0001	—·0001	30·74
5	37	—·0251	—·0877	·0000	—·0000	—·0000	—·0000	36·78

The roots of the equation $\dfrac{d}{d\mu} P_n^m(\mu)=0$

TABLE XIII

$$\theta=\frac{\pi}{12},\ m=0$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	14·5	—·1244	·0336	·0008	—·0009	·0003	·0002	14·12
2	26·5	—·0691	·0178	·0000	—·0001	·0000	·0000	26·29
3	38·5	—·0504	·0121	·0000	—·0000	·0000	·0000	38·35
4	50·5	—·0365	·0092	·0000	—·0000	·0000	·0000	50·39
5	62·5	—·0296	·0074	·0000	—·0000	·0000	·0000	62·41

TABLE XIV

$$\theta=\frac{\pi}{12},\ m=-1$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'_1a_2	n
1	20·5	—·0888	—·0698	—·0005	—·0003	—·0003	—·0002	19·88
2	32·5	—·0565	—·0434	—·0000	—·0000	—·0001	—·0000	32·11
3	44·5	—·0414	—·0315	—·0000	—·0000	—·0000	—·0000	44·22
4	56·5	—·0327	—·0271	—·0000	—·0000	—·0000	—·0000	56·28
5	68·5	—·0270	—·0204	—·0000	—·0000	—·0000	—·0000	68·32

TABLE XV

$$\theta=\frac{\pi}{12},\ m=-2$$

k	ξ	a_1	a_2	a_3	$a_1a'_1$	$a_1a'_2$	a'^1a_2	n
1	26·5	—·0691	—·2676	—·0051	—·0001	—·0007	—·0006	25·06
2	38·5	—·0504	—·1829	—·0009	—·0000	—·0002	—·0002	37·58
3	50·5	—·0365	—·1388	—·0000	—·0000	—·0000	—·0000	49·83
4	62·6	—·0296	—·1119	—·0000	—·0000	—·0000	—·0000	61·96
5	74·5	—·0248	—·0939	—·0000	—·0000	—·0000	—·0000	74·03

PHYSICS

On the Influence of the Finite Volume of Molecules on the Equation of State

BY

Megh Nad Saha, M.Sc. and Satyendra Nath Basu, M.Sc.

It is well known that the departure of the actual behaviour of gases from the ideal state defined by the equation $p=\frac{Nk\theta}{v}$ is due to two causes:—(1) the finiteness of the volume of the molecules, (2) the influence of the forces of cohesion, *i.e.* the attractive forces amongst the molecules. Van der Waals was the first to deduce an equation of state in which all these factors are taken into account; according to van der Waals, we have

$$p=\frac{Nk\theta}{v-b}-\frac{a}{v^2} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where $b=8\times$ volume of the molecules, a defines the forces of cohesion.

In all subsequent modifications of this equation (Clausius, Dieterici, or D. Berthelot), the changes which have been proposed all relate to the influence of the cohesive forces; the part of the argument dealing with the finiteness of molecular volumes is generally left untouched.

But it has been found that the results of experiments do not agree with the predictions of theory if we regard a and b as absolute constants. Accordingly it has been proposed to regard both a and b as functions, of volume and temperature.*

But before proceeding to these considerations, it is necessary to scrutinize whether the influence of finite molecular volumes is properly represented by the term b. From theoretical considerations, the conclusion has been reached that this is not the case. The argument is as follows: According to Boltzmann's theory, the entropy

$$S=k\log W+C,$$

* Compare van der Waals, *Proc. Amst.* 1916; Van Laar, *Proc. Amst* vol. xvi, p. 44.

where k= Boltzmann's gas-constant, W = probabilty of the state. Let us now calculate the probability that a number N of molecules originally confined within the volume V and possessing finite volumes, shall be contained in a volume V. Neglecting the influence of internal forces, the probability for the first molecule is $\frac{V}{V_0}$, for the second molecule the probability is $\frac{V-\beta}{V_0-\beta}$, where $\beta = 8 \times$ volume of a single molecule, for when the first molecule is in position, the space enclosed by a concentric sphere of double the radius of the molecule will not be available for the second molecule. The available space is therefore $V-\beta$, whence the probability is $\frac{V-\beta}{V_0-\beta}$. Introducing similar considerations for the rest of the molecules, we have

$$W = \frac{V}{V_0} \cdot \frac{V-\beta}{V_0-\beta} \cdot \frac{V-2\beta}{V_0-2\beta} \cdots \frac{V-\overline{N-1}\beta}{V_0-\overline{N-1}\beta} \quad \ldots \quad (2)$$

We are, of course, neglecting those cases in which partial overlapping of the ragions occupied by two or more molecules occurs; for the number of such cases can at best be a small fraction of the total number. Even cases of actual association do not include these, for in that case, two discrete molecules become merged into one, without their outer surfaces being actually in contact.

From the relations $\quad S = k \log W + C$

and $$\left(\frac{\partial S}{\partial V}\right)_e = \frac{p}{\theta}$$

we can easily verify that

$$p = -\frac{k\theta}{\beta} \log \frac{V-N\beta}{V}$$

$$= -\frac{R\theta}{2b} \log \frac{V-2b}{V} \;(R=Nk) \quad \ldots \quad \ldots \quad (3)$$

As a first approximation, when b is small compared to v, we obtain $p = \frac{Nk\theta}{v}$ (Boyle-Charles-Avogadro Law), and as a second approximation we obtain

$$p = \frac{Nk\theta}{v-b} \text{ (van der Waals correction).}$$

We also note that

$$pV = Nk\theta.\ \frac{x}{1-e^{-x}},\ \text{where } x = \frac{\beta p}{K\theta}\ \ \ldots\qquad \ldots\ (4)$$

To account for the influence of internal forces, we multiply, following the lead of Dieterici, the above expression (3) by $e^{-\frac{a}{N\theta kv}}$, a having the same significance as before.

From this equation of state, we can easily verify the following results for the critical point:

Critical volume, $\qquad V_c = \dfrac{2e}{e-1}\,b = 3{\cdot}166b,$

$$K = \frac{Nk\theta_c}{p_c V_c} = 3{\cdot}513.$$

The corresponding values of V_c from the van der Waals and the Dieterici equations are ($3b$, $2b$) respectively, and of

$$k \text{ are } \left(\frac{8}{3} = 2{\cdot}66, \frac{e^2}{2} = 3{\cdot}695\right) \text{ respectively.}$$

As a matter of fact, for the simpler gases, the value of k' obtained in this paper agrees better with the experimental results than the Dieterici value $\dfrac{e^2}{2}$; we have for oxygen* $k=3{\cdot}346$, for nitrogen† $k=3{\cdot}53$, for argon‡ $k=3{\cdot}424$, for xenon§ $k=3{\cdot}605$. We need not consider the van-der Waals value $\dfrac{8}{3}$, for it fails entirely.

The most serious drawback to Dieterici's equation is, according to Prof. Lewis (*vide* Lewis's Physical Chemistry, vol. ii, p. 117) that it makes b or the limiting volume $= \dfrac{V_c}{2}$, while the limiting volume, obtained by the extrapolation of Cailletet-Mathias mean density line to

* Mathias and K. Onnes, *Proc. Amst.* Feb. 1911.

† Berthelot, *Bull de la Soc. France de Phys.* 167 (1901)

‡ Mathias, Onnes, and Crommelin, *Proc. Amst.* 1913, p. 960, vol. xv.

§ Paterson, Cripps, Whytlaw-Gray, Proc. Roy. Soc. Lond. A. lxxxvi. p. 579 (1912).

the temperature $\theta=0°$ K is about $\frac{V_c}{4}$. The value of b obtained in this paper, *viz.*, $\frac{V_c}{3 \cdot 16}$ therefore agrees better with this value.

It is yet premature to predict what influence this investigation will have on the speculations concerning the variability of the volume of molecules with temperature. A more detailed investigation dwelling upon this point, and the application of the formula (4) to Amagat's (pv, p) curves, will be communicated shortly. Meanwhile we point out that the factor $e^{-\frac{a}{Nk\theta v}}$ has been introduced into the expression for 'p' only as a provisional measure, though it is considered that this step, though not quite exact, is one in the right direction. In the next paper an attempt will be made to introduce energy in to probability calculations.

We may note here that in several papers in the Amsterdam Proceedings *vide* vol. xv., p. 240 *et seq.*), Dr. Keesom of Leyden had also made attempts to deduce the equation of state from Boltzmann's entropy principle. But, in the expression (2) for W, he introduces, before differentiation, an approximation in which terms up to second order in $\frac{b}{v}$ are retained only. In this way, he arrives at the van der Waals' from $v-b$ for the influence of finite molecular volumes. In obtaining our present equation of state (4), no such approximation has been made. Recently Prof. Kammerling Onnes* has published the critical data for Hydrogen, which may serve as a test for the theory proposed here.

$$\text{Onnes finds } P_c=12 \cdot 8 \text{ } Atm, \; V_c=\frac{1}{d_c}=\frac{1}{\cdot 0310},$$

$$\theta_c=33 \cdot 18, \; K=3 \cdot 38.$$

This value agrees best with our critical constant.

Sir T. N. Palit College of Science, Calcutta.

* Proc. K. A. W. Amsterdam, 1917, 20·178-184.

ON THE LIMIT OF INTERFERENCE IN THE FABRY-PEROT INTERFEROMETER.

BY

MEGH NAD SAHA.

When a monochromatic source of radiation (for example that given by a vacuum tube when excited by an electric discharge) is examined by a Fabry-Perot interferometer, we obtain bright and narrow rings of maximum intensity separated by wide dark intervals. If the distance between the plates of the interferometer be gradually increased, the maxima gradually decrease in brightness, until we reach a limit where we can no longer distinguish between the maxima and the minima. The theory of this phenomena has been worked out by Lippich, Lord Rayleigh,[1] and Schönrock,[2] and is shown to be due to the fact that the emission centres (in this case the gaseous atoms) being in motion, a sort of Döppler-Fizeau effect is produced in the line of vision of the observer. They have shown that when the pressure is small, the critical distance Δ (or the limit of interference) is connected by the following formula with the wave length λ of light, the temperature T of the tube, and the mass M of the emission centres:

$$\frac{\Delta}{\lambda} = A\sqrt{\frac{M}{T}} \qquad \dots \quad (a)$$

This theorem has been made the basis of a wide series of experiments by Michelson,[3] and the French School[4] of opticians including Fabry, Perot, and Buisson. Among the various problems to which the formula (a) has been applied may be mentioned the following.

(*i*) The temperature of the discharge tube when emitting a monochromatic light.

[1] Lord Rayleigh, Phil. Mag. November, 1915.

[2] Schönrock—Ann. *d.* Physik, 1907 B.d. 22.

[3] Michelson—Astrophysical Journal, 1895, vol. (*ii*), p. 251.

[4] Buisson et Fabry—Journal de Physique, tome *ii*, 1912 p. 441-464.

(*ii*) The temperature of stars and nebulae.

(*iii*) Mass of the emission centres of lines in the spectrum. *Probably the mass of the emission centres of many lines of unknown origin in the solar corona and many nebulae (e.g, $\lambda=5007\ A\ v$) which are attributed to hypothetical elements*[1] *Coronium and Nebulium may be determined by this method*, if experimental difficulties con be overcome [2]

object.

The value of the constant A is of much use in all these investigations, and it is generally deduced from theoretical considerations. While going through the literature on the subject, I found that A is generally calculated from approximate and not altogether satisfactory considerations, though an exact solution is not difficult. *My object in the present communication is to effect this improvement in the theory.* For this, we must t begin with a preliminary scrutiny of the theory of the Fabry-Pero Interferometer.

Theory of the Fabry Perot Interferometer.

The Fabry-Perot interferometer consists of two plane parallel plates of glass, both half-silvered on the inside. If a ray of light is sent through the plates, it undergoes several internal reflections, and at each reflection from either surface, a part issues out. Every incident ray is thus subdivided into a large number of parallel rays. If the angle of incidence is very small, almost normal, as is the case in practice, the number would be infinite. Let us confine our attention to the rays issuing on the side further from the source of light. The parallel rays issuing at some particular angle have got path differences amounting to $2d \cos a$, $4d \cos a$, $6d \cos a$ etc. according as they have suffered double reflexion once, twice, thrice, or, any number of times. When these rays are brought together by a converging lens we shall have the interference phenomena. Thet parallel system is composed of

rays transmitted directly, *i.e.* without reflexion,—this ray can be represented by $E_{\circ} \cos nt$,

rays suffering reflexion twice, fourtimes etc. Since at each double reflexion there is a retardation in phase amounting to $\frac{2\pi\Delta}{\lambda}$ and the intensity is reduced by a fraction f, we can represent

[1] Nicholson—Phil. Mag, 1911, vol. 22, p. 864.

[2] Fabry and Bourget Compte Rendu 1917.

the rays by the equations

$$fE_0 \cos(nt-\delta),\ f^2E_0 \cos(nt-2\delta),\ f^3E_0 \cos(nt-3\delta),$$

where we put $\Delta=2d\cos\alpha,\ \delta=\frac{2\pi\Delta}{\lambda}$.

The resultant ray is now represented by

$$E=E_0\{\cos nt+f\cos(nt-\delta)+f^2\cos(nt-2\delta)+\quad\ldots\ \}$$
$$=E_0\{\cos nt(1+f\cos\delta+f^2\cos 2\delta+\ldots)+\sin nt(f\sin\delta+f^2\sin 2\delta+-)$$
$$=E_0\left[\cos nt\frac{1-f\cos\delta}{1-2f\cos\delta+f^2}+\sin nt\frac{f\sin\delta}{1-2f\cos\delta+f^2}\right]$$

Therefore the intensity $I=I_0\dfrac{1}{1-2f\cos\delta+f^2}$

$$\frac{I_0}{(1-f)^2}\ \frac{1}{1+\frac{4f}{(1-f)^2}\sin^2\frac{\delta}{2}}$$

This is the ordinary theory of the interferometer. The intensities of the maxima and the minima are all in the ratio of $1:1+\frac{4f}{(1-f)^2}$. If we take $f=\cdot75$, this ratio becomes $49:1$, the angular separation being $\alpha=\frac{\lambda}{\Delta}$. If the theory held rigorously, we could observe interferecnce with large values of Δ. But this is not the case. For example in the case of the sodium D_1—line, no interference can be obtained when Δ exceeds $3c.m.$ This is due to the fact that the radiant particles are themselves in motion, and the theory cannot be perfect unless we take account of this fact.

Theory is not exact.

According to Maxwell's distribution Law, the number of particles having their velocity between v and $v+dv$ is $Ae^{-\beta v^2}dv$. The frequency of radiation emitted by these particles is $n(1+v/c)$ where n is the wave frequency of light emitted by particles at rest.

Döppler Effect in the Emiddion centre.

In the expression for retardation in phase, we must therfore replace λ by $\lambda\left(1+\frac{v}{c}\right)$ and write $\frac{2\pi\Delta}{\lambda}\left(1+\frac{v}{c}\right)$ in the place of $\frac{2\pi\Delta}{\lambda}$.

The exact theory.

The intensity of light emitted by molecules having their velocity between $v+dv$, and v is

$$dI = B \frac{e^{-\beta v^2} dv}{1-2f\cos\delta(1+v/c)-f^2}$$

The total intensity

$$I = B\int^{\infty} \frac{e^{-\beta v^2} dv}{1-2f\cos\delta(1+v/c)+f^2}$$

We have by Trigonometry,

$$\frac{1-f^2}{1-2f\cos\delta+f^2} = 1+2f\cos\delta+2f^2\cos 2\delta+ \quad \dots$$

Now, we have $\int e^{-\beta v^2} \sin \frac{n\delta v}{} dv = 0$

$$\int_{-\infty}^{\infty} e^{-\beta v^2} \cos \frac{n\delta v}{c} dv = \sqrt{\frac{\pi}{\beta}}\, e^{-\frac{1}{\beta}\left(\frac{n\delta}{c}\right)^2}$$

We have therefore $I = \frac{B_o}{1-f^2} \sqrt{\frac{\pi}{\beta}} \left[1+2\sum_1^{\infty} f^n \cos n\delta\right]$

Now let I_1 = the maximum value of I, corresponding to $\delta=0$

= the minimum value of I, corresponding to $\delta=\pi$

Then the visibility factor V is, according to Michelson

$$=\frac{I_1-I_2}{I_1+I_2} = \frac{fe^{-\frac{1}{\beta}\left(\frac{2\pi\Delta}{\lambda c}\right)^2} + f^3 e^{-\frac{3^2}{\beta}\left(\frac{2\pi\Delta}{\lambda c}\right)^2} + \quad \dots}{\frac{1}{2}+f^2 e^{-\frac{2^2}{\beta}\left(\frac{2\pi\Delta}{\lambda c}\right)^2} + f^4 e^{-\frac{4^2}{\beta}\left(\frac{2\pi\Delta}{\lambda c}\right)^2} + \quad \dots}$$

Now $\frac{1}{\beta}\left(\frac{2\pi\Delta}{\lambda c}\right)^2$ is of the order 10^8. We can, therefore, safely omit terms containing f^2, f^3 etc.

$$V \text{ is therefore} = 2f e^{-\frac{1}{\beta}\left(\frac{2\pi\Delta}{\lambda c}\right)^2}$$

From the kinetic theory of gases, we have

when m=weight of the radiant atom in grams

ω=weight of the hydrogen stom

M=atomic weight of the radiant gas.

κ=universal gas constant

T =temperature

Then we have, since $-\frac{1}{\beta}\left(\frac{2\pi\Delta}{\lambda c}\right)^2=\log_e\left(\frac{V}{2f}\right)$.

Result.

$$\frac{\Delta}{\lambda}=\left[\frac{c}{\pi}\sqrt{\frac{w}{2\kappa}\log_e\left(\frac{2f}{v}\right)}\right]\sqrt{\frac{M}{T}} \qquad (b)$$

Comparison with Lord Rayleigh's Result.

Lord Rayleight took account of the *first two interfering beams only, but by this he had evidently the Michelson interferometer in his mind.* But I think that when we are applying the result to *the Fabry-Perot interferometer, we should take into account all the infinite number of interfering beams, and the effect of reflexion. This is exactly what has been done in the present communication.*

The exact evaluation of the constant $\frac{c}{\pi}\sqrt{\frac{\omega}{2k}\log_e\frac{2f}{V}}$ cannot be done unless the reflecting power of the plates, and the value of V be known. f will depend upon the silvering of the plates, while V will vary with the observer. Thus Lord Rayleigh takes

the visibility factor equivalent to ·025 while Schonrock takes it equivalent to ·05. Assuming that $V = \cdot 025$, and $f = \cdot 75$

We have $\frac{\Delta}{\lambda} = 1 \cdot 50 \times 10^{6} \sqrt{\frac{M}{T}}$. (c)

while according to Lord Rayleigh $\frac{\Delta}{\lambda} = 1 \cdot 42 \times 10^{6} \sqrt{\frac{M}{T}}$.

As it is, the descrepancy between the two values calculated by two different methods is not much. But for particular observers, the discrepancy may be considerable. It is to be hoped that investigators will take notice of these facts.

On the Mechanical and Electrodynamical Properties of the Electron.

BY

Megh Nad Saha.

The object of the present paper is to extend Minkowski's method* of four-dimensional analysis to the investigation of the mechanical and electrodynamical problems connected with the electron. As is well-known, Minkowski has treated the Principle of Relativity by the method of four-dimensional analysis and we have thereby to abandon two time-honoured concepts of Physics *i.e.*, absolute independence of time and space, and the constancy of mass. The correctness of these two principles is no longer a matter of hypothesis, but is founded on experiments. It is therefore to be hoped that the results of these investigations will be helpful to us for the elucidation of the mechanical and electrical problems connected with the electron, though sometimes difficulty may be encountered in putting proper interpretation on these results.

The notation is the same as that adopted by Minkowski, and for the convenience of the reader, it is explained at the very outset.

1

$(x, y, z, l = ict)$ denotes the space and time co-ordinates of any point in the four-dimensional world—

$$(w_1, w_2, w_3, w_4) = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \left[\frac{u_1}{c}, \frac{u_2}{c}, \frac{u_3}{c}, i \right]$$

denotes the velocity four-vector of the point.

* Minkowski's method of four-dimensional analysis is expounded in two papers (1) Raum und Zeit, published in the Phys. Zeits, and (2) Die Grundgleichungen für die Electro-magnetischen Vorgänge in bewegten Körpern-Gött. Nach 1908. These two papers have been translated by me, and are being published by the Calcutta University.

We put $ds^2=-(dx^2+dy^2+dz^2+dl^2)$; therefore we have

$$(w_1, w_2, w_3, w_4)=\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}, \frac{dl}{ds}\right), \text{ and } -\sqrt{-1}\,(w_1, w_2, w_3, w_4)$$

denote the direction cosines of the four-dimensional tangent to the path of the particle.

We put $(f_{23}, f_{31}, f_{12})=(H_x, H_y, H_z)$, the components of the magnetic field, and $(f_{14}, f_{24}, f_{34})=-i(E_x, E_y, E_z)$ the components of the electric field. Minkowski has shown that f constitutes a six-vector.

$(a_1, a_2, a_3, a_4)=[F, G, H, i\phi]$, are the components of the potential four-vector; (F, G, H) are the vector potentials, ϕ is the scalar potential.

ρ=electrical space-density;

$$\rho\left[\frac{u_1}{c}, \frac{u_2}{c}, \frac{u_3}{c}, \frac{u_4}{c}, i\right]=\rho_0|w_1, w_2, w_3, w_4|$$

are the components of the stream four-vector s; $\rho_0=\rho\sqrt{1-\frac{u^2}{c^2}}$ is known as the rest-density of electricity.

The vector operator $\square=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}+\frac{\partial}{\partial l}\right)$ is known as the lor and the scalar operator $\square^2=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial l^2}\right)$ is known as the generalised D'alembertian.

The equations of electrodynamics can be written in the forms

$$\text{lor } f=s, \qquad \text{lor } f^*=0,$$

$$f=\text{curl } a, \ \square^2 a=-s, \ \square a=0.$$

2

The Scalar and Vector Potentials of moving electron.

Lienard,* and almost simultaneously Wiechert† showed that the scalar and vector potentials are given by the expressions

$$\phi=\frac{e}{r(1-u_r)}, \quad (F, G, H)=\frac{e(u_1, u_2, u_3)}{c\,r(1-u_r)}. \qquad \ldots \qquad \ldots \quad (1)$$

* Lienard, L'Eclairage electrique 16 (1898), pp. 5, 53, 106.

† Wiechert :—Arch. Néerl (2), 5 (1900).

If P be the point at which the potentials are calculated at the time t, and M be the position of the electron at the time t_0, where $MP=c(t-t_0)$, the distance MP is denoted by r and $[u]$ denotes the velocity in the position M, and (u_r) its components in the direction of r.

The formulae are deduced from the theory of retarded potentials and do not involve the principle of relativity.

Sommerfield* has shown that the formulae can also be deduced from the theory of relativity and can be thrown into the compact form

$$\mathbf{a}=\frac{[w]e}{[R\cdot w]}, \text{ R being the four-vector joining the two points.} \quad \dots \quad (2)$$

It is quite clear that the forms (1) and (2) are quite equivalent. $(R\cdot w)$ denotes the scalar product of the two four vectors R and w.

In a paper published elsewhere, it has been shown that from Minkowski's four-dimensional analysis we obtain

$$\mathbf{a}=\frac{e[w]}{P} \quad \dots \quad \dots \quad \dots \quad (3)$$

In this formula, (x, y, z, l) denote the time-space co-ordinates of the electron (A), (w_1, w_2, w_3, w_4) its velocity-components, $(x'\ y'\ z'\ l')$ denote the space-time co-ordinates of the point B at which the potentials are estimated.

P denotes the four-dimensional perpendicular distance of B from the axis of motion of (A); since the direction-cosines of this axis are $-i(w_1, w_2, w_3, w_4)$, we have

$$P^2=(x-x')^2+(y-y')^2+(z-z')^2+(l-l')^2+[(x-x')w_1 + (y-y')w_2+(z-z')w_3-(l-l')w_4]^2.$$

Now if we make the assumption that the time co-ordinates are so chosen that

$$(x-x')^2+(y-y')^2+(z-z')^2+(l-l')^2=0 \quad \dots \quad (4)$$

i.e., $$c^2(t-t')^2=r^2,$$

or $$c(t-t')=r,$$

*Sommerfeld—über die Relativitäts-Theorie, Ann. d. Physik, Vols. 32 and 33.

the formula (3) becomes the same as (2), and therefore (1); also the assumption which we make here about the interval between the time-co-ordinates is identical with the premises of Lienard and Wiechert.

I am not quite certain whether this assumption (4) which is made here is at all essential. I am inclined to think that it is not essential, but necessary only for the interpretation of the result to those three-dimensional beings whose senses are not sharpened enough to enable them to grasp a result expressed in four-dimensional figures.

3

The Electric and Magnetic Fields due to a moving electron.

If a denote the potential four-vector, the components of the six-vector f giving the electric and magnetic fields are given by

$$f = \text{Curl}\ \mathbf{a} = \left| \begin{array}{cccc} \dfrac{\partial}{\partial x'} & \dfrac{\partial}{\partial y'} & \dfrac{\partial}{\partial z'} & \dfrac{\partial}{\partial l'} \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{array} \right.$$

Thus

$$f_{12} = \frac{\partial \mathbf{a}_2}{\partial x'} - \frac{\partial \mathbf{a}_1}{\partial y'} = -\frac{e}{\mathrm{P}^3} \left| w_3\, a_2 - w_2\, a_3 \right|, \text{ etc.}$$

where $\qquad a_1 = \mathrm{P}\dfrac{\partial \mathrm{P}}{\partial x'},\ a_2 = \mathrm{P}\dfrac{\partial \mathrm{P}}{\partial y'}$, etc.

we can easily verify that if we put $c(t-t') = r$, we have

$$\mathrm{H}_x = \frac{e}{r^3\lambda^3}\left[\frac{u_y}{c}(z-z') - \frac{u_z}{c}(y-y')\right],\ \text{where } \lambda = \left(1 - \frac{u_r}{c}\right),$$

$$= \frac{e}{r^3\lambda^3 c}\,[\mathrm{V}\times r].$$

The electric forces are given by

$$f_{14} = -i\,\mathrm{E}_x = \frac{\partial \mathbf{a}_4}{\partial x'} - \frac{\partial \mathbf{a}_1}{\partial l'},$$

$$= -\frac{e}{\mathrm{P}^3}\left[w_1(l-l') - w_4(x-x')\right]$$

$$= \frac{e}{r^3\lambda^3}\left[\left(\frac{\mathrm{V}}{c}(x) - r\right)\right]$$

and generally $E_x = \frac{e}{r^3\lambda^3}\left[(x-x') - \frac{r u_1}{c}\right]$ (6)

These values are widely different and simpler than the values obtained from the older theories, for example, campare the values given by Crehore (Physical Review, July, 1917, p. 148).

The discrepancy is due to the fact that in these older theories, we always assume that the equation

$$(x-x')^2+(y-y')^2+(z-z')^2+(l-l')^2=0$$

is an essential condition. But in performing differentiations with regard to (x', y', z', l'), we here assume that they are quite independent of (x, y, z, l). I am not quite definite as to which of these two standpoints is more in accordance with Minkowski's ideas of time and space. However it is preferable to keep an open mind on this point.

Maxwell's Stresses, Poynting-vector, etc.

Minkowski has shown that if we multiply f by its own matrix, we obtain a matrix

$$ff= \begin{matrix} S_{11}-L & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22}-L & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33}-L & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44}-L \end{matrix}$$

where

$$S_{11}=\tfrac{1}{2}[f_{23}^2+f_{34}^2+f_{42}^2-f_{12}^2-f_{13}^2-f_{14}^2]$$

$$S_{12}=[f_{13}f_{32}+f_{14}f_{42}],$$

$$L=\tfrac{1}{2}[f_{23}^2+f_{34}^2+f_{42}^2+f_{12}^2+f_{13}^2+f_{14}^2],$$

and the matrix $\frac{1}{4\pi}\begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{vmatrix} = \begin{vmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{vmatrix},$ denotes the

Maxwellian stresses, $i(S_{14}, S_{24}, S_{34})$ denotes the components of the Poynting-vector, and S_{44} is the energy function. We have generally

$$X_x = \frac{1}{8\pi}\left[f_{23}^2 + f_{34}^2 + f_{42}^2 - f_{12}^2 - f_{13}^2 - f_{14}^2\right]$$

$$X_y = \frac{1}{4\pi}\left[f_{13}\, f_{32} + f_{14}\, f_{42}\right]$$

etc.

Now on the standpoint taken up by me, it is quite easy to calculate these quantities. It can be shown that

$$X_x = \frac{e^2}{8\pi P^6}\left[-a^2(1+2w_1^2) + a_1^2\right],\quad X_y = \frac{e^2}{4\pi P^6}\left[-w_1 w_2 a^2 + a_1 a_2\right],$$

The Poynting-vector (X_t, Y_t, Z_t) (7)

$$= \frac{e^2}{4\pi P^6}\left[(-a_1 a_4 + w_1 w_4 a^2), (-a_2 a_4 + w_2 w_4 a^2), (-a_3 a_4 + w_3 w_4 a^2)\right] \quad (7')$$

and the energy function $S_{44} = L_t = \dfrac{e^2}{8\pi P^6}\left[-a^2(1+2w_4^2) + a_4^2\right],$ (7'')

where $a_1 = P\dfrac{\partial P}{\partial x},\ a_2 = P\dfrac{\partial P}{\partial y},\ \ldots\ \ldots$

and $a^2 = a_1^2 + a_2^2 + a_3^2 + a_4^2 = P^2.$

5

The law of attraction between two moving electrons.

We can now proceed to find out the attraction which one moving electron exerts upon another moving electron.

According to Lorentz's theorem the components of the force acting on an electron (A) moving in any electromagnetic field are

$$\begin{aligned} X &= e[\phantom{w_1 f_{21} +{}} w_2 f_{12} + w_3 f_{13} + w_4 f_{14} \\ Y &= e[w_1 f_{21} \phantom{{}+ w_2 f_{12}} + w_3 f_{23} + w_4 f_{24} \\ Z &= e[w_1 f_{31} + w_2 f_{32} \phantom{{}+ w_3 f_{13}} + w_4 f_{34} \end{aligned} \quad (8)$$

and we can also add the fourth or the time-component

$$L = -\frac{ie}{\beta c}[Xu_1+Yu_2+Zu_3],\ \beta=\sqrt{1-\frac{u^2}{c^2}}$$ which is proportional to the rate at which work is done by the moving charge. We have

$$L = e[w_1 f_{14}+w_2 f_{24}+w_3 f_{34}]. \qquad (8')$$

In this case, the field is due to a second electron, (charge e', position $x'\ y'\ z'\ l'$, velocity components $(w_1'\ w_2'\ w_3'\ w_4')$.

According to the last section, the potential four-vector

$$\mathbf{a} = \frac{e'[w']}{\cdots},\ \text{where}$$

$$\begin{aligned}P^{2\prime} = (x-x')^2+(y-y')^2+(z-z')^2+(l-l')^2+[(x-x')w_1'\\ +(y-y')w_2'+(z-z')w_3'+(l-l')w_4']^2,\end{aligned}$$

$$f = \text{Curl } \mathbf{a},$$

$$\begin{aligned}X = ee'\Big[w_2\Big\{\frac{\partial}{\partial x}\Big(\frac{w_2'}{P'}\Big)-\frac{\partial}{\partial y}\Big(\frac{w_1'}{P'}\Big)\Big\}+w_3\Big\{\frac{\partial}{\partial x}\Big(\frac{w_3'}{P'}\Big)\\ -\frac{\partial}{\partial z}\Big(\frac{w_1'}{P'}\Big)\Big\}+w_4\Big\{\frac{\partial}{\partial x}\Big(\frac{w_4'}{P'}\Big)-\frac{\partial}{\partial l}\Big(\frac{w_1'}{P'}\Big)\Big\}\end{aligned}$$

$$\begin{aligned}= ee'\Big[\frac{\partial}{\partial x}\Big(\frac{w_1w_1'+w_2w_2'+w_3w_3'+w_4w_4'}{P'}\Big)-\Big(w_1\frac{\partial}{\partial x}+w_2\frac{\partial}{\partial y}\\ +w_3\frac{\partial}{\partial z}+w_4\frac{\partial}{\partial l}\Big)\frac{w_1'}{P'}\Big]\end{aligned}$$

Now putting $\Phi = ee'(w_1w_1'+w_2w_2'+w_3w_3'+w_4w_4')/P'$, (9)

find that $X = \dfrac{\partial\Phi}{\partial x}-\dfrac{d}{ds}\Big(\dfrac{\partial\Phi}{\partial w_1}\Big)$, (10)

where $\frac{\partial}{\partial x}$ denotes differentiation in which x is only explicitly involved, similarly with $\dfrac{\partial}{\partial w_1}=\dfrac{\partial}{\partial\frac{dx}{ds}}$

$\frac{d}{ds}=\left(w_1\frac{\partial}{\partial x}+w_2\frac{\partial}{\partial y}+w_3\frac{\partial}{\partial z}+w_4\frac{\partial}{\partial l}\right)$, as is easily seen. We have similarly,

$$Y=\frac{\partial\Phi}{\partial y}-\frac{d}{ds}\left(\frac{\partial\Phi}{\partial\frac{dy}{ds}}\right),\quad Z=\frac{\partial\Phi}{\partial z}-\frac{d}{ds}\left(\frac{\partial\Phi}{\partial\frac{dz}{ds}}\right),$$

$$L=\frac{\partial\Phi}{\partial l}-\frac{d}{ds}\left(\frac{\partial\Phi}{\partial\frac{dl}{ds}}\right). \qquad (10')$$

We can say that Φ is the kinetic-potential of the electron (A) in the field of the electron (B). Similarly if Φ_1 denotes the kinetic-potential of the electron (B) in the field of (A),

$$\Phi_1=ee'(w_1w_1'+w_2w_2'+w_3w_3'+w_4w_4')/P, \qquad (11)$$

$$P^2=(x-x')^2+(y-y')^2+(z-z')^2+(l-l')^2+[(x-x')w_1+(y-y')w_2+(z-z')w_3+(l-l')w_4]^2.$$

and we have similarly $X'=\frac{\partial\Phi'}{\partial x'}-\frac{d}{ds'}\left(\frac{\partial\Phi}{\partial\frac{dx'}{ds'}}\right)$, etc, ... (12)

Let us now interpret the results in three dimensions. We have

$$X=\frac{ee'\beta'^2(x-x')}{r^3\lambda^3\beta}\left(1-\frac{uu'\cos\theta}{c^2}\right)-\frac{ee'\beta'^2}{r^2\lambda^3\beta c}\left(1-\frac{u_r}{c}\right)u'_1, \qquad (13)$$

where

$$\beta=\sqrt{1-\frac{u^2}{c^2}},\ \beta'=\sqrt{1-\frac{u'^2}{c^2}},\ \lambda=\left(1-\frac{u_r}{c}\right)$$

In three dimensions, the forces are equivalent to a force

$$\frac{ee'\beta'^2}{r^2\lambda^3\beta}\left(1-\frac{uu'\cos\theta}{c^2}\right), \qquad (14)$$

in the direction of the line joining the two points, and a force

$$\frac{ee'\beta'^2}{r^2\lambda^3\beta c}\left(1-\frac{u_r}{c}\right)u', \qquad (15)$$

in the direction of the velocity of the second or the attracting point.

We thus perceive that the force which comes out in a very simple form in four-dimensions takes a very complicated form in three-dimensions.

$$\text{The kinetic-potential } \Phi = \frac{ee'\left(1-\dfrac{uu'\cos\theta}{c^2}\right)}{r\left(1-\dfrac{u'_r}{c}\right)}, \tag{16}$$

$$=\frac{ee'}{r}\left[1+\frac{u_r'}{c}+\frac{u_r'^2}{c^2}-\frac{uu'\cos\theta}{c^2}+\ \dots\right]$$

The kinetic-potential is practically coincident with the kinetic-potential assumed by Clausius in order to find out the law of attraction between two moving charges of electricity. Clausius has shown that this kinetic-potential leads us to the celebrated electrodynamic laws of Ampére. A short resumé of the work done in this connection is given below for the purpose of comparison. The problem was first enunciated by Gauss in the year 1835, and was called by him the fundamental keystone of electiodynamics.*

(1) Gauss:—The forces are the derivatives with regard to (x, y, z)

of the potential function $\phi=\dfrac{ee'}{r}\left[1-\dfrac{3}{2c^2}\dfrac{d^2r}{dt^2}\right]$. (16')

(2) Weber takes the potential function $\phi=\dfrac{ee'}{r}\left[1-\dfrac{1}{c^2}\left(\dfrac{\partial r}{\partial t}\right)^2\right]$ (16'')

Both of these forms have been long discredited. Latter writers have pointed out that the force cannot be simply the derivations with regard to (x, y, z) of some potential function, but are the Lagrangian derivatives of a certain kinetic-potential. We give the forms of this kinetic-potential according to different investigators.

(1) Clausus $\Phi=\dfrac{ee'}{r}\left(1-\dfrac{uu'\cos\theta}{c^2}\right)$ [1881] (16''')

* For the literature on the subject, see Maxwell, Electricity and Magnetism Vol. 2, Chap. XXXIII, and J. J. Thomson, Application of dynamics to problems of Physics and Chemistry, pp. 35, et. esq.

where u and u' are the velocities of the two electrons, and θ is the angle between their lines of motion.

(2) J. J. Thomson (1882). $\phi=-\frac{ee'}{r}\left(1-\frac{\mu uu'\cos\theta}{c^2}\right)$, μ=magnatic permeability, (here $\mu=1$.

Crehore has calculated the forces components according to J. J. Thomson's theory (Phil. Mag. 1915). He finds that the forces are equivalent to

$F_1=\frac{ee'}{r^2}$ ... a repulsion along the line joining the centres.

$F_2=\frac{ee'}{r^2 r^2}uu'\cos\theta$—an attraction along the line joining the centres.

$F_3=\frac{ee'}{r}\dot{u}'$—a force in a direction opposite to the acceleration of the second charge.

$F_4=ee'u'\frac{d}{dt}\left(\frac{1}{r}\right)$ — a force in a direction opposite to the motion of the second charge. (16'')

(3) Sommerfeld (Ann. d. Phys. Vols. 32 and 33, über die Relativitäts-theorie) has also calculated the ponderomotive forces, assuming that the value of the potential four vector $\mathbf{a}=\frac{e[w]}{[Rw]}$, and using the condition $(x-x')^2+(y-y')^2+(z-z')^2+(l-l')^2=0$, in course of differentiation.

5 Equations of Motion of the Electron.

Minkowski deduces the equations of motions of a ponderable particle by means of a variational process in which the function $\int m_0 c^2 ds$. where $ds^2=-(dx^2+dy^2+dz^2+dl^2)=c^2dt^2\left(1-\frac{u^2}{c^2}\right)$ is used instead of the three-dimentional function $\int T\,dt$.

He obtains

$$m_0 c^2 \frac{d^2 x}{ds^2} = X,\ m_0 c^2 \frac{d^2 y}{ds^2} = Y,\ m_0 c^2 \frac{d^2 z}{ds^2} = Z,\ m_0 c^2 \frac{d^2 l}{ds^2} = L. \qquad (17)$$

Now we have $X = e[w_2 f_{12} + w_3 f_{13} + w_4 f_{14}]$, according to Lorentz's theorem. we have also

$$\frac{d^2 x}{ds^2} = \left(w_1 \frac{\partial}{\partial x} + w_2 \frac{\partial}{\partial y} + w_3 \frac{\partial}{\partial z} + w_4 \frac{\partial}{\partial l} \right)(w_1)$$

$$= w_1 \frac{\partial}{\partial x} \left[\frac{1}{2}(w_1^2 + w_2^2 + w_3^2 + w_4^2) \right] + w_2 \left(\frac{\partial w_1}{\partial y} - \frac{\partial w_2}{\partial x} \right)$$

$$+ w_3 \left(\frac{\partial w_1}{\partial z} - \frac{\partial w_3}{\partial x} \right) + w_4 \left(\frac{\partial w_1}{\partial l} - \frac{\partial w_4}{\partial x} \right) \Big],$$

$$= -(w_2 \Omega_{12} + w_3 \Omega_{13} + w_4 \Omega_{14}),\ \text{Putting } \Omega_{hk} = \frac{\partial w_h}{\partial x_k} - \frac{\partial w_k}{\partial x_h}. \qquad (18)$$

Hence we have the four equations.

Putting $\mu = c^2 m_0 / e$

$$\begin{array}{llll}
 & w_2(f_{12}+\mu\Omega_{12}) & +w_3(f_{13}+\mu\Omega_{13}) & +w_4(f_{14}+\mu\Omega_{14})=0 \\
w_1(f_{21}+\mu\Omega_{21}) & & +w_3(f_{23}+\mu\Omega_{23}) & +w_4(f_{24}+\mu\Omega_{24})=0 \\
w_1(f_{31}+\mu\Omega_{31}) & +w_2(f_{32}+\mu\Omega_{32}) & & +w_4(f_{34}+\mu\Omega_{34})=0 \\
w_1(f_{41}+\mu\Omega_{41}) & +w_2(f_{42}+\mu\Omega_{42}) & +w_3(f_{43}+\mu\Omega_{43}) & =0
\end{array}$$

Of these, only three are independent; the fourth can be deduced from the first three.

We have indentically

$$0 = \begin{vmatrix}
 & f_{12}+\mu\Omega_{12}, & f_{13}+\mu\Omega_{13}, & f_{14}+\mu\Omega_{14} \\
f_{21}+\omega_1\Omega_{21} & & f_{23}+\mu\Omega_{23}, & f_{24}+\mu\Omega_{24} \\
f_{31}+\omega_1\Omega_{31}, & f_{32}+\omega_2\Omega_{32} & & f_{34}+\mu\Omega_{34} \\
f_{41}+\omega_1\Omega_{41}, & f_{42}+\omega_2\Omega_{42}, & f_{43}+\omega_3\Omega_{43} &
\end{vmatrix}$$

i.e.

$$(f_{12}+\mu\Omega_{12})\ (f_{34}+\mu\Omega_{34}) + (f_{33}+\mu\Omega_{33})\ (f_{14}+\mu\Omega_{14})$$
$$+ (f_{32}+\mu\Omega_{31})\ (f_{24}+\mu\Omega_{24}) = 0. \qquad (19)$$

* Minkowski-Die Grundgleichungen für die Elektromagnetischen Vorgänge in bewegten Körpern-Math. Annalen, Vol. 68—Anhang—Mechanik.

The condition is evidently satisfied if

$$-\mu=\frac{f_{12}}{\Omega_{12}}=\frac{f_{23}}{\Omega_{23}}=\frac{f_{31}}{\Omega_{31}}=\frac{f_{14}}{\Omega_{14}}=\frac{f_{24}}{\Omega_{24}}=\frac{f_{34}}{\Omega_{34}} \tag{19'}$$

But this relation is not correct as the following variational process shows.

Let (X, Y, Z, L) represent the components of the force four-vector at any point, which is subjected to a virtual deplacement δx, δy, δz, δl.

Then $\delta W = X\delta x + Y\delta y + Z\delta z + L\delta l.$

$$i.e., \text{ if we call } W=\frac{\partial A}{\partial s},\ A=\int W ds,$$

$$\delta A=\int \delta W ds=\int (X\delta x+Y\delta y+Z\delta z+L\delta l)ds$$

$$=e\int[f_{12}(dy\delta x-\delta y dx)+f_{23}(dz\delta y-\delta z dy)+f_{31}(dx\delta z-\delta x dz)$$
$$+f_{14}(dl\delta x-\delta l dx)+f_{24}(dl\delta y-\delta l dy)+f_{34}(dl\delta z-\delta l dz)].$$

Now the function $\int m_0 c^2 ds$ can also be subjected to a variational process. Since

$$ds=w_1 dx+w_2 dy+w_3 dz+w_4 dl.$$

we find $$\delta\int m_0 c^2 ds=m_0 c^2\int[\Omega_{12}(dy\delta x-\delta y dx)+\Omega_{13}(dz\delta x+\delta z dx)$$
$$+\Omega_{23}(dz\delta y-\delta z dy)+\Omega_{14}(dl\delta x-\delta l dx)+\Omega_{24}(dl\delta y-\delta l dy)$$
$$+\Omega_{34}(dl\delta z-\delta l dz)].$$

But it is not possible to equate to zero the coefficients of the six-components $(dx\delta y—\delta x dy)$ of the area-six-vector $(dS\times\delta s)$, for though $(\delta x$, δy, δz, $\delta l)$ represent an arbitrary displacement, (dx, dy, dz, dl) is not so but represent the actual displacements. We have, therefore, to collect the coefficients of $(\delta x, \delta y, \delta z, \delta l)$ and put them separately equal to zero. In this way we obtain

$$-\frac{m_0 c^2}{e}=\frac{f_{12}\omega_2+f_{13}\omega_3+f_{14}\omega_4}{\Omega_{12}\omega_2+\Omega_{13}\omega_3+\Omega_{14}\omega_4}=\frac{f_{21}\omega_1+f_{23}\omega_3+f_{24}\omega_4}{\Omega_{23}\omega_1+\Omega_{23}\omega_3+\Omega_{24}\omega_4}$$

$$=\frac{f_{31}\omega_1+f_{32}\omega_2+f_{34}\omega_4}{\Omega_{31}\omega_1+\Omega_{32}\omega_2+\Omega_{34}\omega_4}=\frac{f_{41}\omega_1+f_{42}\omega_2+f_{43}\omega_3}{\Omega_{41}\omega_1+\Omega_{42}\omega_2+\Omega_{43}\omega_3}, \tag{20}$$

which are simply another from of the Minkowskian equations (17)

$$m_o c^2 \frac{d^2 x}{ds^2} = X, \ m_o c^2 \frac{d^2 y}{ds^2} = Y, \ m_o c^2 \frac{d^2 z}{ds^2} = Z, \ m_o c^2 \frac{d^2 l}{ds^2} = L \quad (A')$$

for

$$\frac{d^2 x}{ds^2} = -(\omega_2 \Omega_{12} + \omega_3 \Omega_{13} + \omega_4 \Omega_{14}), \quad \text{etc.}$$

Difficulty is encountered here about the interpretation of the terms Ω_{12}, Ω_{13} etc. in three dimensions Ω is evidently a six-vector being the four-dimensional curl of the velocity four vector.

The components $[\Omega_{23}, \ \Omega_{31}, \ \Omega_{12}]$ are evidontly connected with rotations $\left[\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \ \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \ \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right]$, $[\Omega_{14}, \ \Omega_{24}, \ \Omega_{34}]$

are connected with the accelerations $\left[\frac{d^2 x}{dt^2}, \ \frac{d^2 y}{dt^2}, \ \frac{d^2 z}{dt^2}\right]$ but the exact interpretation in three-dimensions has not yet been obtained. *We can style Ω as the* **acceleration six-vector.**

We shall calculate the path of the electron in a few well-known cases, according to the method expounded in the present paper.

1. Under a uniform magnetic field (in the direction of the x-axis), suppose the particle is initially projected with velocity v, in the direction of the y-axis. Then all the f's except $f_{23} = 0$ and $f_{23} = H$.

Then either from the equations of motion (17) or from the curl-equations (20) it can be at once seen that

$$\frac{dl}{ds} = w_4 = \text{constant}, \ \frac{dx}{ds} = w_1 = \text{constant} = 0.$$

$\therefore \ w_2^2 + w_3^2$ remains constant *i.e.*, speed remains constant.

$$\text{Now} - \frac{He}{m_o c^2} = \frac{\partial w_3}{\partial y} - \frac{\partial w_2}{\partial z} = \frac{1}{c\sqrt{1 - \frac{u^2}{c^2}}} \left[\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right],$$

now $\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z} = \frac{v}{r}$, where r=radius of the path.

$$\therefore \left(\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\right) u = \frac{eHr}{c}, \text{ or } mv = \frac{eHr}{c}, \; m=\text{transverse-mass}.$$

2. Under a uniform electrostatic field.

Let the field be in the direction of x. Then all the f's except f_{14} which $=-iE_x$, are zero. Let the initial velocity be v in the direction of the y-axis. We shall write X instead of E_x.

Then we can easily see from the equations that

$$w_3 = \text{constant} = 0, \; w_2 = \text{constant} = \frac{v}{\sqrt{1-\frac{v^2}{c^2}}}$$

and $\frac{d^2x}{ds^2} = n\frac{dt}{ds}$, $\frac{d^2t}{ds^2} = -n\frac{dx}{ds}$, where $n = -\frac{}{m_0c^2}$.

From this we find

$$x = \frac{w_{40}}{n}(1-\cos ns), \; l = \frac{w_{40}}{n}\sin ns, \; y = w_{20}\, s, \; z=0.$$

The orbit is given by

$$x = \frac{w_{40}}{n}\left(1-\cos\frac{ny}{w_{20}}\right) = \frac{nw_{40}}{2(w_{20})^2}\, y^2$$

The radius of curvature at the origin

$$r = \frac{(w_{20})^2}{nw_{40}} = \frac{v^2}{c^2}\cdot\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,\frac{m_0c^2}{Xe}$$

In other words, $mv^2 = Xer$,

a result which is well known.

On Maxwell's Stresses

BY

Megh Nad Saha.*

1. Maxwell† has shown that the mechanical action between two electrical systems at rest can be accounted for by assuming the existence of certain stresses distributed over a surface completely enclosing one of the systems. If ψ be the potential at any point due to the whole system, the X-component of the mechanical force on one of the systems can be shown to be

$$F_x = \frac{1}{4\pi}\iiint \frac{\partial \psi}{\partial x} \nabla^2 \psi \, dx\, dy\, dz, \quad . \quad . \quad . \quad (1)$$

where the integration extends over the space occupied by the first system.

2. If the force be really due to the presence of stresses on a surface enclosing the first system, we have

$$F_x = \iint X_n dS = \iint (lX_x + mX_y + nX_z) dS, \quad . \quad . \quad (2)$$

where X_x, X_y, X_z &c. ... are the various surface-tractions, and (l, m, n) are the direction cosines of the normal to the surface.

By transforming expression (2), we obtain

$$F_x = \iiint \left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} \right) dx\, dy\, dz.$$

Since $\frac{1}{4\pi}\frac{\partial \psi}{\partial x} \nabla^2 \psi = \frac{\partial}{\partial x}\left[\frac{1}{8\pi}\left\{\left(\frac{\partial \psi}{\partial x}\right)^2 - \left(\frac{\partial \psi}{\partial y}\right)^2 - \left(\frac{\partial \psi}{\partial z}\right)^2\right\}\right]$

$$+ \frac{\partial}{\partial y}\left[\frac{1}{4\pi}\frac{\partial \psi}{\partial x}\cdot\frac{\partial \psi}{\partial y}\right]$$

$$+ \frac{\partial}{\partial z}\left[\frac{1}{4\pi}\frac{\partial \psi}{\partial x}\cdot\frac{\partial \psi}{\partial z}\right].$$

* Reprinted from the Phil. Mag. March, 1917.

† Maxwell, 'Electricity and Magnetism,' vol. i., chap. v.

we have, putting $\frac{\partial\psi}{\partial x}=X,\ \frac{\partial\psi}{\partial y}=Y,\ \frac{\partial\psi}{\partial z}=Z,$

$$\iiint\left\{\frac{\partial}{\partial x}\left[X_x-\frac{1}{8\pi}(X^2-Y^2-Z^2)\right]+\frac{\partial}{\partial y}\left[X_y-\frac{1}{4\pi}XY\right]\right.$$

$$\left.+\frac{\partial}{\partial z}\left[X_z-\frac{1}{4\pi}XZ\right]\right\}dx\,dy\,dz=0. \quad . \quad . \qquad (3)$$

Maxwell concludes from this that a system of stresses

$$X_x=\frac{1}{8\pi}(X^2-Y^2-Z^2),\quad Y_y=\frac{1}{8\pi}(Y^2-Z^2-X^2),$$

$$Z_z=\frac{1}{8\pi}(Z^2-X^2-Y^2),\quad X_y=\frac{1}{4\pi}XY,\quad Y_z=\frac{1}{4\pi}YZ,\quad Z_x=\frac{1}{4\pi}ZX, \quad . \quad (4)$$

distributed over the surface S, accounts for the mechanical action quite satisfactorily, and therefore provides a concrete physical representation of the mechanism of electrostatic action.

3. But the expressions (4) are not complete solutions of the integral equation (3). Maxwell * himself points out that they can at best be regarded as a first step towards the solution of equation (3). Many investigators, including Sir J. J. Thomson †, have pointed out that æther cannot possibly be at rest under these stresses. Lorentz ‡ goes so far as to say that the stresses are simply mathematical fictions, which can be conveniently utilized for the calculation of radiation pressure and other allied phenomena. The object of the present paper is to show that the stresses cannot account for the energy of electrification, if the medium is to be regarded as an elastic solid.

4. The energy of a charged system can be expressed as a volume integral,

$$W=\frac{1}{8\pi}\iiint\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2+\left(\frac{\partial\psi}{\partial z}\right)^2\right]dx\,dy\,dz. \quad . \qquad (5)$$

Maxwell § states that the quantity W may be interpreted as the energy in the medium due to the distribution of stresses; but the

* *Loc. cit.* p. 165 *et seq.*

† *Loc. cit.* p. 165, footnote.

‡ 'Theory of Electrons,' p. 31.

§ 'Electricity and Magnetism,' p. 165.

statement is not proved. The only rational meaning which we can attach to this assertion is that the energy of electrification arises from the elastic displacement of æther particles. I am not aware whether any other interpretation has been or can be given to Maxwell's statement, but it has generally been taken in this sense, though Maxwell himself is rather vague on the point. We should naturally expect that energy calculated on this understanding would lead to the expression (5), but that this is not the case will be presently shown.

5. If u, v, w be the elastic displacements of a particle of the dielectric medium, the energy of deformation or the strain-energy function is

$$W' = \iiiint_{\text{initial state}}^{\text{final state}} \rho(X\delta u + Y\delta v + Z\delta w)\, dx\, dy\, dz$$

$$+ \iiint_{\text{initial state}}^{\text{final state}} (X_\nu \delta u + Y_\nu \delta v + Z_\nu \delta w) dS. \tag{6'}$$

and this can be shown to be equivalent to

$$\frac{1}{2}\iiint (X_x e_{xx} + Y_y e_{yy} + Z_z e_{zz} + X_y e_{xy} + Y_z e_{yz} + Z_x e_{zx}) dx\, dy\, dz.$$

Assuming the æther to be isotropic and to behave as an elastic solid, we can put

$$\begin{matrix} e_{xx} \\ \\ e_{zz} \end{matrix} \quad \frac{1}{E} \begin{bmatrix} 1 & -\sigma & -\sigma \\ -\sigma & 1 & -\sigma \\ -\sigma & -\sigma & 1 \end{bmatrix} \begin{bmatrix} X_x \\ Y_y \\ Z_z \end{bmatrix}$$

and $$e_{xy} = \frac{X_y}{\mu}, \quad e_{yz} = \frac{Y_z}{\mu}, \quad e_{zx} = \frac{Z_x}{\mu}.$$

Then, after some calculation, the strain-energy function comes out to be

$$W' = \frac{1}{2}\,\frac{3(1+2\sigma)}{\epsilon(1+\sigma)} \iiint \left(\frac{R^2}{8\pi^2}\right)^2 dx\, dy\, dz. \tag{6}$$

It will thus be seen that if the stresses are really existent, and if they are amenable to the ordinary laws of elasticity,

the strain-energy function, or the energy of elastic deformation of the medium, is $\frac{3}{2}\frac{(1+2\sigma)}{\epsilon(1+\sigma)}\left(\frac{R^2}{8\pi^2}\right)^2$ per unit volume. But this is very different from the theorem that the energy density per unit volume is $\left(\frac{R^2}{8\pi}\right)$, which is derived from electrostatic principles.

6. Since nothing definite is known about the elastic constants of æther, we cannot draw any conclusion from (6) about the energy distribution in æther. Maxwell's stresses are thus seen to fail to account for the energy of electrification, on the understanding that the medium behaves like an elastic solid.

7. It is well known that the energy-distribution theorem is proved on the basis of the empirical laws of electrostatics. No use is made of the stresses. The result is purely analytical, and says that if energy is distributed all over space as a continuous function with volume density $\frac{R^2}{8\pi}$, the total energy will come out to be the same as the total energy of electrification. The distinction between Maxwell's view of energy distribution as due to stresses (in the sense we have interpreted it) and the actual case can be better brought out if we adopt the following modified method of proving the energy-distribution theorem. Suppose we have an electrical system consisting of charged surfaces, and particles in a given configuration. The energy of electrification will be the same in whichever way we may bring about the final configuration. Suppose that, to start with, the charges and the charged surfaces were all at an infinite distance, and the given configuration is brought about by properly moving the charged surfaces and other discreet electrified particles. Then the energy of electrification is

$$W=\Sigma\int e\delta V,$$

$$=\Sigma\iint\sigma\left(\frac{\partial V}{\partial x}\delta x+\frac{\partial V}{\partial y}\delta y+\frac{\partial V}{\partial z}\delta z\right)dS,$$

$$+\Sigma\iiint\rho\left(\frac{\partial V}{\partial x}\delta x+\frac{\partial V}{\partial y}\delta y+\frac{\partial V}{\partial z}\delta z\right)dx\,dy\,dz,$$

where σ is the surface density of electricity on a charged surface, and ρ is the volume density. Since $\frac{\partial V}{\partial x}$ is the x-component of electrical force on a surface, $\sigma \frac{\partial V}{\partial x}$ is the x-component of mechanical action per unit surface. Similarly, $\rho \frac{\partial V}{\partial x}$ is the x-component of mechanical force per unit volume of electrified particles. We can therefore put

$$W = \Sigma \iint (X_r \delta x + Y_r \delta y + Z_r \delta z)\, dS + \Sigma \iiint (X\delta x + Y\delta y + Z\delta z)\, dV.$$

8. Comparing this expression for energy with the expression (6)

$$W' = \iint (X_r \delta u + Y_r \delta v + Z_r \delta w) dS + \iiint \rho\, (X\delta u + Y\delta v + Z\delta w)\, dV. \quad (6)$$

we see that in the present case (X_r, Y_r, Z_r) are the components of surface-tractions on a charged surface, and (X, Y, Z) are the body-forces on electrified particles. The existence of these forces can be experimentally demonstrated, and they exist only in regions occupied by electricity; elsewhere they are nil. The energy of electrification is derived from the work done in the actual displacements $(\delta x, \delta y, \delta z)$ of these charged regions towards each other. On the other hand, (X_r, Y_r, Z_r) in (6) are the tractions on a surface enclosing some of the charged regions, and $(\delta u, \delta v, \delta w)$ are their elastic displacements. We may by special assumption identify the two systems of surface-tractions and body-forces, but the actual displacements $(\delta x, \delta y, \delta z)$ and the elastic displacements $(\delta u, \delta v, \delta w)$ cannot be identified in any way. The two expressions represent fundamentally different quantities.

9. The fact that radiant energy would exert a definite amount of pressure on material surfaces was first predicted by Maxwell on the hypothesis of dielectric stresses. Now that radiation pressure is an experimental fact, it has been supposed by some physicists that Maxwell's stresses must have a material existence. But it is well known that radiant energy can be deduced independently of the stresses. Bartoli has shown that the pressure of radiant energy can be deduced

from thermodynamic principles. Planck* has deduced it from electrodynamical principles, assuming that the perfect reflector is a super-conductor of electricity. This is an ideal limiting case of the experimental fact that good conductors of electricity are also good reflectors of radiant energy. The electric vector accompanying a ray of light gives rise to a finite charge on the surface of the super-conductor and a finite current within the conductor. The charge exerts a negative pressure on the surface, while the current, in presence of the field of the magnetic vector accompanying the ray, produces a mechanical force in the contrary direction. The resultant of the two, when averaged statistically, yields the radiation pressure. How far these theories are consistent with the theory of stresses may form a subject for interesting investigations.

My best thanks are due to Prof. D. N. Mallik for his kind help and encouragement.

* Planck, *Wärmestrahlung*, second edition, p. 49 *et seq.*

On a New Theorem in Elasticity

BY

Megh Nad Saha, M.Sc.*

1. The equations of motion of an elastic system are†

$$\left.\begin{aligned}\rho\ddot{u}&=\rho X+\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}\\ \rho\ddot{v}&=\rho Y+\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}\\ \rho\ddot{w}&=\rho Z+\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}\end{aligned}\right\} \quad (1)$$

Multiplying the equations by u, v and w, and adding, we have,

$$\because u\ddot{u}=\tfrac{1}{2}\frac{d^2}{dt^2}(u^2)-\dot{u}^2,$$

$$\frac{\rho}{2}\frac{d^2}{dt^2}(u^2+v^2+w^2)-\rho(\dot{u}^2+\dot{v}^2+\dot{w}^2)$$

$$=\rho(Xu+Yv+Zw)+u\left(\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}\right)$$

$$+v\left(\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}\right)+w\left(\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}\right) \quad (2)$$

Now multiplying by $(dx\,.\,dy\,.\,dz\,.\,dt)$, and integrating we have,

$$\because \text{since} \iiint u\left(\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial x}\right)dx\,dy\,dz=\iint uX_n\,dS$$

$$-\iiint\left(X_x\frac{du}{\partial x}+X_y\frac{du}{dy}+X_z\frac{du}{dz}\right)d\Omega,$$

$$\iiiint_{t=0}^{t=\tau}\frac{\rho}{2}\frac{d^2}{dt^2}(u^2+v^2+w^2)\,dt\,d\Omega-\iiiint_{t=0}^{t=\tau}2T\,dt\,d\Omega$$

$$=\iiiint_{t=0}^{t=\tau}\rho(Xu+Yv+Zw)\,dt\,.\,d\Omega$$

* Reprinted from the J. A. S. B., Oct., 1918.

† Love's Elasticity, page 83.

$$+\iiint_{t=0}^{t=\tau}(X_n u+Y_n v+Z_n w)dS\,dt-\iiiint_{t=0}^{t=\tau} 2W\,d\Omega\,dt,$$

where

$$2W=X_x e_{xx}+Y_y e_{yy}+Z_z e_{zz}+X_y e_{xy}+X_z e_{xz}+Y_z e_{yz} \tag{3}$$

Denoting by $\overline{T}$ the time-average of kinetic energy per unit volume, and by $\overline{W}$ the time-average of the potential energy per unit volume, we have

$$\iiint(\overline{W}-\overline{T})\,d\Omega=\frac{1}{2\tau}\iiiint_{t=0}^{t=\tau}\rho\,(Xu+Yv+Zw)\,dt\,.\,d\Omega$$

$$+\frac{1}{2\tau}\iiint_{t=0}^{t=\tau}\rho\,(X_n u+Y_n v+Z_n w)\,dS\,dt$$

$$-\frac{1}{\tau}\iiint\left[\frac{\rho}{4}\,\frac{d^2}{dt^2}\,(u^2+v^2+w^2)\,\right]_{t=0}^{t=\tau}d\Omega \tag{4}$$

2. If we now take a closed volume Ω and $\overline{\overline{W}}$, $\overline{\overline{T}}$ denote the average values over time as well as over space, we shall have

$$\overline{\overline{W}}-\overline{\overline{T}}=\frac{1}{2\Omega\tau}\iiiint_{t=0}^{t=\tau}\rho\,(Xu+Yv+Zw)\,d\Omega\,dt$$

$$+\frac{1}{2\Omega\tau}\iiint_{t=0}^{t=\tau}(X_n u+Y_n v+Z_n w)\,dS\,dt \tag{5}$$

Since if τ be sufficiently large, the function $\frac{d}{dt}\,(u^2+v^2+w^2)$ will have the same value at the beginning and end of the process if the motion be vibratory, for then τ will contain a large number of periods.

3. The analogy of theorem (5) to Clausius's* Virial theorem is quite evident. According to the virial theorem, we have

$$-T=\tfrac{1}{2}\Sigma\Sigma xX+yY+zZ,$$

where T=kinetic energy of the number of particles within unit volume,

(X, Y, Z)=force components on the particle which occupies the point (x, y, z).

4. A number of interesting applications are at once suggested.

* *Vide* Jeans' Dynamical Theory of Gases, Second Edition, page 141.

Suppose the motion to be vibratory. Then if the body forces be nil, the average kinetic energy will be equivalent to the average potential energy if

(*i*) the surface tractions be nil, or constant, as in the case of the vibration of a supported rod, or plate with free ends,

(*ii*) the surface displacement be zero,

(*iii*) if part of the surface be under zero or constant stress and part under varying stress with no surface displacement (*e.g.*, the case of a clamped rod, or string stretched between two points).

These theorems are of course well known, and can be deduced in other ways.

On the Pressure of Light

BY

Megh Nad Saha, M.Sc. and Sudhakar Chakravarti, B.Sc.*

The object of the present paper is to describe a simple apparatus by means of which the pressure of light can be easily demonstrated, and qualitatively measured with the entire elimination of all sorts of disturbing effects. The materials required are not difficult to procure, and are readily available in all well-equipped laboratories.

We wish first to give a short history of the subject and a short sketch of the theory.† As early as the seventeenth century Kepler supposed that light exerted a pressure on surfaces on which it is incident. The hypothesis was proposed for explaining the tails of comets.

With the rise of Newton's corpuscular theory of light, the pressure no longer remained a guess, but could be deduced from that theory. An elaborate series of experiments for detecting the pressure was instituted by De Mairan (1754), and later on by Du Fay (1756), but the results were entirely negative Later on, the failure of these experiments were used as arguments against the validity of the corpuscular theory of light.

But interest in the subject was again revived when Maxwell,‡ in the year 1873, predicted that even on the basis of the electro-magnetic theory of light, radiant energy should exert a pressure on a surface on which it is incident. But the amount of pressure is extremely small. It can be shown that if light consists of unidirectional rays, the pressure amounts to $\frac{1}{c}$ (Amount of radiant energy falling on unit surface per unit of time, measured in absolute units), where c velocity of light, and the surface is a perfectly absorbing one, *e.g.*, a surface coated with lamp-black.

If the surface on which the light is incident be perfectly reflecting, the pressure is just double. But if, on the other hand, the surface be transparent (*e.g.*, glass), there will be no pressure at all, or more ac-

* Reprinted from the J. A. S. B., Oct., 1910.

† For the historical part, see Lebedew, Ann.-d. Phys., Bd. 6, page 433; and Nichols and Hull, Phys. Rev., 1903.

‡ Maxwell, Electricity and Magnetism, Vol. II, page 792.

curately a very small amount of pressure depending on the small amount of reflection from the glass-surface.

The occurrence of the term c in the denominator makes the pressure extremely small. Let us take for example the pressure exerted by solar light. The amount of energy which is delivered by the sun on unit surface placed normally to the rays of the earth is equivalent to 2·4 calories per minute. The pressure therefore

$$=\frac{2{\cdot}4\times4{\cdot}2\times10^{7}}{3\times10^{10}\times60\times981}\text{ gms. weight}={\cdot}56\times10^{-7}\text{ gms. weight per }(cm)^{2}.$$

By using the arc, or a very high candle power filament lamp (1500 wt/$\frac{1}{2}$ wt. for example), and by concentrating the light by means of a lens of large aperture, *the pressure can be increased to about 100 times.* But still it is extremely small.

It was for demonstrating the pressure of light that Crookes* was led to invent his famous "radiometer." As is well known, this consists of a delicate cross of glass or mica vanes suspended on a pivot and enclosed within a glass cylinder from which air can be pumped off at will. The alternate faces of the vanes are covered with lamp-black. When light falls on the vanes it begins to rotate rapidly about the axis.

Crookes was inclined to explain this motion as being due to the pressure of radiant energy, but Johnstone-Stoney showed that the effect observed was rather spurious, and exceeded theoretical pressure by at least 10^5 times.‡ He showed that the effect was really due to the unequal heating of the two sides of the vanes.

Zöllner§ tried to observe the effect by another arrangement. Two thin discs of silvered or blackened glass, or metal, were suspended at the ends of the horizontal arm of a thin cross of glass-rods and the whole was suspended by means of glass fibres within a closed vessel, from which air can be pumped out at will. A galvanometer mirror is attached to the vertical part, with its plane at right angles to the plane of the vanes. But with light incident on the vanes, the deflection observed was very irregular, and sometimes was completely in the wrong direction.

But in spite of repeated failures to detect the pressure of radiation, theoretical investigation had, in the meantime, been advanced so far that it was not possible to deny its existence.

* Phil. Trans. 1874, Vol. 164, p. 501.

† Pogg. Ann. Bd. 160, p. 154, 1877 (suggested by Maxwell).

‡ Bartoli, Nuovo Cimento, 15, p. 195, 1883.

§ Hull, Phys. Rev., May, 1905.

We have seen that the pressure of light was deduced by Maxwell from the electromagnetic theory of light, by using an argument involving the assumption of pressures and tensions across and along tubes of force. But Bartoli showed in 1877 that the pressure could also be deduced by means of thermodynamic reasoning involving only the two laws of thermodynamics, and was in amount just the same as is obtained from Maxwell's theory. Bartoli's argument being based on the surer basis of thermodynamics, seemed to carry conviction in all quarters about the real existence of the pressure.

The long-expected pressure was at last observed by Lebedew, and almost simultaneously by Nichols and Hull in 1901, by different modifications of Zöllner's unsuccessful experiment.

Lebedew's method was to replace the rather thick glass vanes by means of very thin platinum foils (diameter 5 mm., thickness ·C2 mm.) whereby any difference of temperature on the two sides is instantly equalised. The radiometer action is directly proportional to the difference of pressure on the two sides, and the pressure of gas within the vessel. Lebedew reduced the pressure to about 1/20000th of a mm. and was almost able to eliminate the radiometer action, and verify the pressure qualitatively within about 20°/₀ of the theoretical pressure.

The early experiments of Nichols and Hull were undertaken in order to investigate the different disturbing effects in the apparatus of Zöllner. They found that the total disturbing effect is the resultant of the following:—

(*i*) the radiometer action—due to the unequal heating of the two sides of the vane;

(*ii*) convection currents—due to the rush of air towards the parts warmed by the passage of the pencil of rays;

(*iii*) a rocket action—due to the escape of particles of gas from the surface of the vanes when these are heated by the incident light.

By a series of elaborate investigations extending over three years, Nichols and Hull were able to get rid of these effects. They found that the convection effect could be reduced by making the vanes exactly vertical, for then the flow of air becomes tangential to their surface. The rocket action, and the radiometer action were found to balance at a pressure of 16·5 mm., and deflections were therefore observed with this pressure in the vessel. The vanes were of thin glass with one face silvered; for further information on the point reference should be made to the original paper.

Finally, Hull evolved out an arrangement by means of which the disturbing effects could be entirely eliminated. The silvered side of a thin cover-glass was placed in contact with the blackened side of a similar glass and the whole was enclosed by means of two other thin glasses, as shown in the figure. Two such cells were mounted upon the opposite ends of the torsion arm which was suspended by means of a thin quartz fibre within a glass cylinder. When light falls on the vanes, the two sides are of course unequally heated. But as the air on the two sides is enclosed within a glass cylinder, it forms one single system with the glass vessel—action and reaction being equal, the radiometer action is entirely eliminated.

We have found the extra glass cell to be redundant. The silvered sides of two thin cover-glasses were put one upon the other and connected to each other by means of a trace of Canada-balsam on the fringes. Similarly, we prepared a lamp-blacked surface. We have thus in these vanes very thin films of totally reflecting and totally absorbing material enclosed within equal thicknesses of glass on either side. When light, previously filtered of all rays capable of heating glass, is allowed to fall on one of the vanes, say the silvered one, the glass surface is not at all heated by the passage of the rays, which have been previously passed through sufficiently thick glass lenses. The two sides of the film are instantly raised to the same temperature (because they are extremely thin and there being equal thicknesses of glass on the two sides, they are equally heated by conduction). Thus the radiometer action is entirely eliminated.

It will be thus seen that in our arrangement we have combined the arrangements of Lebedew as well as Hull's method, without the additional encumbrance of extra glass cells.

Description of the Apparatus.

The vanes were suspended on the opposite arms of the torsion balance: *vide* fig. 2. (*m*) is a galvanometer mirror placed at right angles to the plane of the vanes, with a small piece of steel on its back. (B) is a small brass weight for steadying the balance. The whole is suspended by means of a glass fibre and enclosed within a bell-jar which is connected to a pump and a manometer. The deflection is observed from the excursion of a spot of light reflected from (*m*) in the usual lamp and scale arrangement. The dimensions are

Diameter of the cover glasses	=1·8 cm.
Thickness of the cover glasses	= ·083 mm.
Weight of the silvered vane	= ·105 gm.

Weight of the lamp-blacked vane $=$ ·128 gm.
Length of the arm $=$ ·2 cm.
Weight of B $=$ ·5 gm.

The pressure within the bell-jar is reduced to about 1 to 2 cms. of mercury. It is extremely important that the joints should be all air-tight, for the slightest leakage of air may produce disastrous effects. After pumping out we allowed the apparatus to stand for 3 days in order to be sure that it was quite air-tight The vanes should be placed symmetrically just about the centre of the glass vessel, otherwise currents of air which are set up in the vessel by the passage of rays and turned off by the sides of the vessel may produce disturbing effects. These effects become smaller, the smaller the pressure inside the vessel.

Theory of the Apparatus :—

The equation of motion of the vanes is given by—

$$I\frac{d^2\theta}{dt^2}+k\frac{d\theta}{dt}+\mu\theta=L \qquad \text{(i)}$$

where (I) moment of inertia of the system about the fibre. k viscosity factor, μ is the torsional coefficient, θ=angle of rotation, L moment of the force of pressure about the axis of rotation (*i.e.*, the fibre).

The solution is
$$\left(\theta-\frac{L}{\mu}\right)=Ae^{\frac{k}{2I}t}\cos(nt+\beta) \qquad \text{(ii)}$$

where
$$n^2=\frac{\mu}{I}-\frac{k^2}{4I^2}. \qquad \text{(iii)}$$

After a sufficiently large time the deflection should become steady if the disturbing causes are entirely absent. Let a denote this steady deflection.

Now $L=pl$, where p=total pressure (or thrust) on the surface and l=distance of the centre of the disc from the axis of rotation. The light should be concentrated on the centre of the disc. Let a be the steady deflection. Then

$$p=\frac{\mu a}{l}.$$

The constant μ is obtained from observations of the free period of oscillation of the system.

From (ii) we see that $\frac{\mu}{I}=n^2+\frac{k^2}{4I^2}$.

Now $n=\frac{2\pi}{T}$, and $\frac{k^2}{4I^2}=\left(\frac{\beta}{T}\right)^2$, where β=logarithmic decrement of the amplitude.

$$\therefore \frac{\mu}{I}=\left(\frac{2\pi}{T}\right)^2+\left(\frac{\beta}{T}\right)^2=\frac{1}{T^2}(4\pi^2+\beta^2). \qquad \text{(iv)}$$

Now "I" can be easily calculated from the weight and the dimensions of the system. μ can therefore be easily calculated from formula (iv).

In our experiment $l=2{\cdot}65$ cm. and $\mu=6{\cdot}27\times10^{-6}$, so that a deflection of 1 mm. at a distance of 1 metre corresponded to a total pressure of

$$\frac{6{\cdot}27\times10^{-6}}{2{\cdot}65}=2{\cdot}36\times10^{-6} \text{ dynes.}$$

The time period was 32 seconds and the logarithmic decrement was $\beta={\cdot}310$, and $I=1{\cdot}67$ units.

Measurement of Energy.

Owing to lack of means at our disposal the amount of energy falling upon the surface could not be properly measured. Lebedew allowed the light to fall on a copper calorimeter placed in the same position as the vanes, and the amount of energy absorbed was obtained by noting the rise in temperature of the calorimeter within a given period of time.

Nichols and Hull's method was more ingenious. A thin disc of silver of the same size as the vane was coated with lampblack. Two holes were bored on the sides through which a copper-constantan couple passed. The other end of the couple passed through a sensitive galvanometer. This apparatus was previously standardised by putting it in different baths. The light was allowed to fall on the disc for some time and the rise in temperature was obtained from the throw of the galvanometer.

The source of light in Lebedew and Hull's experiment was an arc which as is well known is very unsteady. In our early experiments we used the arcs but in the latest experiment the source of light was a (3000 c.p.) Tungsten filament lamp supplied by Messrs. Westinghouse & Co. The light from this source is very steady. The lamp was placed in a horizontal position (*i.e.*, with its filament in a vertical circle) at a distance of 50 to 70 cms. from the diaphragm which contained a short focus lens of 6·5 cms. aperture. By adjusting the lens the filament was completely focussed on the vanes. An upper limit to the amount of energy falling on the vane per second can be thus obtained. By

means of an ammeter we found that the lamp consumed a current 6·6 amps. under a pressure of 220 volts. The amount of energy passing through the lens and focussed on the vanes is therefore given by

$$\frac{220 \times 6{\cdot}6 \times 10^7}{4\pi(d)^2} \pi (3{\cdot}25)^2 \text{ ergs. per sec.}$$

The whole pressure on the silvered surface is therefore

$$\frac{E}{c}(1-\epsilon)(1+\rho)$$

where c-velocity of light and ϵ=fraction of energy absorbed by and reflected from glass surfaces (lens and containing vessels) and ρ=fraction of energy reflected from the silvered face.

Results of Observations.

In our preliminary blank experiment with the arc, we found that for the period for which the arc remains steady, the deflection remains quite steady and follows very faithfully the fluctuations of the arc. When the positive pole was focussed the deflection observed was generally 3 to 4 times the deflection for the negative pole. When all the precautions above mentioned were taken, the deflection was found to be always in the right direction. When the filament lamp was used as the source of light, all irregularities due to the variation of the source of light vanished. As soon as the light is struck, the spot of light slowly creeps up towards the new position of equilibrium about which it oscillates in accordance with the equation (i).

Ultimately the oscillation dies away and the spot becomes quite steady, which could be maintained for 15 minutes (we did not try to keep the spot steady for a greater length of time because the tungsten filaments, being kept in a horizontal position, are gradually deformed on account of their plasticity at the high temperature within the lamp.

In one set of experiments one of the vanes was silvered while the other consisted of two clear pieces of microscopic cover-glass. We found that when light was allowed to fall on the clear glass surface there was practically no deflection. In another set of experiments one of the vanes was silvered and the other was lamp-blacked. It was found that generally if the source of light was not too intense, the deflection of the black surface was approximately one half of that of the silvered one. If the source of light was very intense, so much heat was absorbed that

the junctions (which were all of shellac) melted off. Quantitative experiments were therefore impossible with that surface.

One of the results of our quantitative experiments is given below:—

Mean deflection (mean of several experiments) ... $=28{\cdot}5$ Divns.

Distance of the scale from the mirror ... $=100$ cm.

Distance "d" of the plane of the filament from the diaphragm. ... ... $=73$ cm.

Therefore the upper limit of the total theoretical pressure (without allowing for absorption or reflexion) is equal to

$$2\times\frac{6{\cdot}6\times220\times10^{7}\times(3{\cdot}25)^{2}}{4\times73^{2}\times3\times10^{10}}=4{\cdot}8\times10^{-4}\text{ dynes.} \qquad \text{(A)}$$

The pressure calculated from deflections is equal to

$$2{\cdot}3\times10^{-5}\times14{\cdot}25=3{\cdot}33\times10^{-4}\text{ dynes.} \qquad \text{(B)}$$

The observed pressure is about 70 per cent. of expression (A), which is the pressure calculated on the supposition that the whole amount of energy given out by the filament is freely transmitted by the various glass media, and is totally reflected by the silvered surface. As a matter of fact, none of these assumptions is correct. If T is the fraction of total energy transmitted by thick glass, and ρ be the reflecting power of a silver glass-surface the actual pressure should be

$$P_0\frac{T}{2}(1+\rho)(1-\epsilon)$$

where P_0 is the quantity (A).

According to the experiments of Rubens and Hagen* $\rho=90{\cdot}5\%$; unfortunately no data is available for the transmission co-efficient, but on account of the preponderance of rays of short wave length in the spectrum of the light from a tungsten filament, it cannot be less than 80%.

Considering these facts, we are probably justified in asserting that the agreement between observed and theoretical values is at least qualitatively quite good. On a future occasion we hope to return to the problem of a rigorous quantitative determination of total incident energy.

In conclusion, we beg to record our best thanks to Prof. C. V. Raman, and the teaching staff of the University College of Science, for the interest they have taken in the work; and to Mr. N. Basu, B.Sc., for much useful help.

* Obtained by extrapolation from the data of Rubens and Hagen on the supposition that the maximum emission of energy from a tungsten filament is at 1 μ [Kohlrausch, Praktische Physik, Tabellen].

On Radiation Pressure and the Quantum Theory. A Preliminary Note *

BY

Megh Nad Saha, D.Sc.

After the prediction by Maxwell of the existence of the pressure of radiant energy on the basis of his theory of stresses and strains in æther, other ways of arriving at the same result have been found by Bartoli (thermodynamical), Poynting (flow of momentum along a ray of light) and Larmor (electro-magnetic wave-theory of light). A review of these methods shows that they are all statistical, *i.e.*, the result holds only when the surface encountered by radiation is large compared with the wave of light and is thickly set with matter.

Schwarzschild and more recently Nicholson,[1] and Klotz[2] have worked out, on the basis of the continuous theory, the value of the radiation-pressure, when the size of the obstructing mass is gradually decreased, ultimately being reduced to the scale of the wave-length of light. In this case, the effect of the repulsing light pressure gradually preponderates over any gravitative force to which the particle may be subject, but at the same time, it appears that there is a limit to this process of reduction. If the particle be too small, it is no longer capable of acting as a barrier to the advancing light-waves, and consequently experiences no radiation-pressure. It appears from these investigations that for particles of the molecular size (radius 10^{-8} cm) the effect of light-pressure is totally evanescent.

But this conclusion from the old continuous theory is rather contradictory to the requirements of astrophysics—for in order to explain tails of comets, and other astrophysical phenomena (such as solar prominences, corona) which take place on the surface of luminous heavenly bodies, we have to assume the existence of certain repulsive forces[3] (levity) acting on the ultimate gaseous molecules and thus reducing the gravitational attraction on them. But a still stronger ground for rejecting the conclusion is furnished by the experimental demonstration by Lebedew [4] of the existence of radiation pressure on

* Reprinted from the Astrophysical Journal, U. S. A., 1919.

molecules of absorbing gases like CO_2, methane, propane, etc. It may thus be taken for granted, inspite of the failure of the continuous theory that molecules do really suffer a radiation-pressure, which in the aggregate conforms to Maxwell's Law.

Prof. Wood [5] is inclined to the opinion that the gas-molecule may be capable of stopping the radiation by resonance, and may thus experience a radiation pressure, but precisely what is meant by stoppage of radiation by resonance is not clear. An explanation of the existence of radiation-pressure on molecules is furnished when we apply the quantum theory in the place of the old continuous theory of light. Instead of assuming that 'light' is spread continuously over all points of space, let us suppose that they are localized in pulses of energy $h\nu$ (ν=frequency of light, h=Planck's universal radiation constant). Let this pulse encounter a molecule m and be absorbed by it. Then in doing so the molecule will be thrust forward with an impulsive momentum of $\frac{h\nu}{c}$, (c=velocity of light); for we may suppose the pulse to have the mass $\frac{h\nu}{c^2}$, and the momentum $\frac{h\nu}{c}$, the absorption of the pulse by the molecule may be taken as a case of inelastic impact, the whole momentum being communicated to the molecule. The velocity with which the molecule will move forward $=\frac{h\nu}{cm}$.

Let us consider the effect of the absorption of a pulse of the hydrogen light corresponding to the line $H\alpha$ by the hydrogen atom. The velocity imparted at each kick of light,

$$v=\frac{h\nu}{cm}=60 \text{ c. m. per second}$$

taking $h=6{\cdot}54\times10^{-27}$

$$\frac{c}{\nu}=\lambda=6{\cdot}563\times10^{-5},\quad m=\frac{1}{6{\cdot}062\times10^{23}}\text{ gm.}$$

The velocity is rather a small quantity (compared to the orbital velocity of the molecules) but it should be remembered that it is really an impulsive velocity and is of the nature of an acceleration.

The total velocity acquired by a hydrogen atom per second will depend upon the number of kicks of light it experiences per second, and provided this is sufficiently great, the velocity acquired may rise to enormous values. But *a priori* we cannot say what this number will amount to, without a preliminary examination of the physical conditions.

This conclusion explains Lebedew's results which cannot be explained by the continuous theory, and at the same time offers a general explanation of the radiation pressure. The pressure $= \frac{1}{c}\Sigma\Sigma h\nu$ where the summation extends over all the pulses absorbed in unit time, within unit area. It thus equals AI, where I = intensity of light, A = fraction absorbed. The aggregate effect remains unchanged, but it is now supposed to be concentrated on a few active molecules, the inactive molecules remaining uneffected.

The explanation offered closely resembles Einstein's explanation of the velocity of emission of the photo-electrons. According to Einstein, when a pulse of light ($h\nu$) falls upon an atom it is instantly absorbed, and goes to increase the energy of the system. Consequently certain of the electrons of an atomic system acquire a velocity which is greater than the critical velocity required for retaining these electrons in their orbit. Let A be the energy required for detaching an electron from a parent atom. Then the velocity of escape is given by the law

$$\Sigma \tfrac{1}{2} mv^2 = h\nu - \Sigma A$$

The maximum velocity occurs when only one electron is emitted. Then $\frac{1}{2}mv^2 = h\nu - A$.

Actual experiments by Millikan [8] have established the truth of the law quantitatively. Besides the phenomena is instantaneous whatever be the intensity of the light. Now this feature is not capable of explanation by the continuous theory of absorption. Campbell [9] has found that in certain cases, the continuous theory requires that the atom must be illuminated for at least 45 minutes, before it can acquire the energy sufficient for the emission of the electron, while actually the emission takes place in less than $\frac{1}{1600}$ of a second after illumination.

Let us therefore see how the number of kicks of light experienced by the hydrogen atom or molecule varies with the existing circumstances. The number will clearly depend upon the following factors:—(*i*) the density of pulses of light in the region traversed by the molecule, (*ii*) the time of retention by the molecule or the atom of the capacity for the absorption of light. We shall first take up the second point. Hydrogen under ordinary circumstances does not absorb its characteristic radiation (represented by the Balmer lines) as has been demonstrated by the repeated failures of the experiments for obtaining the reversal of the H—lines. But the experiments of Ladenburg and Loria [6] have thrown a new light on the cause of these failures; they find that hydrogen is capable of absorbing its characteristic radiation only when it is in an active state, *i.e.*, when it is in a state of luminescence. This conclusion is also borne out by the theoretical investigations of Bohr, for according to his theory, an H—line is emitted when the attendant electron leaps from orbit (3) to orbit (2), while in the natural state the electron is at orbit (1).

We may symbolically express the idea in the following manner—

Natural state (when inactive)—⊙ radius (1), (state 1).

State (when emitting the Balmer lines)—⊙ radius (4), (state 2).

In order that an H-atom may absorb a Balmer line, it must be to start with, at state (2).

We may thus take it for granted that the H-atoms which absorb the Balmer lines are not the ordinary H-atoms, but an active modification of it, the electron being at orbit (2), instead of at orbit (1). When light corresponding to any line of the Balmer spectrum traverses a mass of hydrogen, it is only the active particles which will absorb this light, and be subjected to the impulsive kicks of this light.

Taking it for granted that an active molecule suffers a discontinuous kick of light given by the formula (1) in the process of absorption, let us see how it will behave when placed in a field of radiation. To visualise matters, we shall take an active H-atom moving near the photosphere of the Sun. The H-atom, if active to start with, will pick out from the continuous spectrum the pulse corresponding to $H\alpha$ or $H\beta$, and will be thrust forward with an instantaneous velocity of 60-81 cms. per second. It is true that as the particle emits light, it suffers an equal recoil opposite the direction

of emission. But it should be borne in mind that the emission does not take place in any specified direction, but in any direction according to the law of chance,* while the pulses which are absorbed come from a specified direction, *viz.*, the centre of the Sun. Hence if the particle continues active for a sufficient length of time, the H-atom may ultimately acquire a velocity exceeding the critical velocity of $6'12 \times 10^7$ cms. per sec., (the velocity required for the escape of a particle from the gravitational influence of the Sun). The precise velocity which a particle acquires, depends upon a large number of unknown factors. (*i*) The intensity of the field of radiation—the influence of this factor is to a certain extent known—the density of pulses varies as the intensity of light, and therefore follows the inverse square law. (*ii*) The persistence of the activity of the H-atom, or rather if the activity be lost, the quickness with which it is regenerated. (*iii*) The actual proportion of active particles in any region.

Nothing is known about the second and the third factors, consequently it is not possible to work out a quantitative theory of the effect of radiation-pressure on the expulsion of the molecules. But the general consideration show that radiation-pressure on molecules may be out of all proportion to their actual sizes. It also shows that the radiation-pressure exerts a sort of sifting action on the molecules, driving the active ones radially outwards along the direction of the beam. The cumulative effect of the pulses may be sufficiently great to endow the atoms with a large velocity—the velocity with which the tops of solar prominences are observed to shoot up.

The velocity of the red prominences are sometimes found to be as high as $8'34 \times 10^7$ cm.

The solar prominences have sometimes been explained on the assumption that they are due to the convection of hot masses of vapour from the solar photosphere, which after reaching the

* We may refer to the experiments of Wood, Strutt, and Dunoyer on the lateral emission by a column of Na-vapour, which is traversed by *strong D_1 or D_2-light.* (*Vide* Strutt, Proc. Roy. Soc., London, September, 1919). It will be interesting to see whether a column of Na-vapour, when traversed by a *strong beam of white light*, emits the D_1, D_2 light laterally. If this is found to be the case, it will constitute a very strong experimental proof in support of the views advanced in this paper.

atmosphere, are supposed to expand adiabatically and develop the large velocities with which the prominences are observed to shoot up. But Pringsheim and Strutt[10] (Monthly Notices of the R.A.S.) have pointed out several insuperable difficulties in the way of the acceptance of this hypothesis, including the deduction that the maximum velocity obtainable from adiabatic expansion is less than $\frac{1}{60}$ of the velocity with which the prominences are observed to shoot forward ($8{\cdot}34 \times 10^7$ cm.). Strutt has suggested that some unknown forces of electrical origin may be the cause of these large velocities, but even granting that the electrical fields exist in the Sun, it is difficult to see how these can act upon the luminous hydrogen particles which are most probably uncharged. According to the hypothesis put forward in this paper, the effect of radiation-pressure on the separate particles are altogether disproportionate to the dimensions of the particles, and may causeit to be endowed with a 'levity' long sought for in the explanation of the prominences, the corona, and other solar phenomena.

Attention may be called to a comprehensive paper by D. Brunt (Month. Not. R.A.S.—1912-13, p. 568) who has shown that neither of the three theories of the equilibrium of the solar atmosphere (Isothermal, adiabatic, or radiative) can account for an atmosphere extending to the observed height of the solar atmosphere. The results of the spectro-heliographic observations are distinctly unfavourable to Julius's theory of anomalous dispersion (*vide* Astrophysical Journal, papers by Hale, St. John, and Adams).

This hypothesis presents the problem of the radiative equilibrium of the solar atmosphere in a new light.

These ideas may be applied to the explanation of the tails of comets. The tails of comets are undoubtedly caused by some sort of repulsive action exerted by solar light, but since on the older theory, the effect was found evanescent on particles of the molecular size, the tail was supposed to consist of some sort of cosmic dust. But the spectroscopic examination of the light from the tails shows that they consist, at least partly, of luminous gases (CO, CO_2)[11]. Now the explanation is quite easy, if the considerations advanced in this paper hold. As the comet approaches the Sun, more and more pulses of light from the Sun traverse the nucleus and the coma.

Light pulses of suitable frequency are picked up by the gaseous particles, which thus gradually gain in velocity in a direction opposite to that of sunlight. The cumulative effect of the absorbed pulses may endow the particle with a velocity sufficient for its escape from the main mass of the cometary matter, and form into the tail.

It is hoped to develop these ideas further in a future communication.

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On Selective Radiation Pressure and the Radiative Equilibrium of the Solar Atmosphere

BY

Megh Nad Saha, D.Sc., *Premchand Roychand Scholar.*

§ 1

In a paper recently communicated to the Astrophysical Journal, an attempt has been made to prove that the quantum theory affords a basis for the existence of radiation-pressure on atoms and molecules.[1] It is well-known that according to the older continuous theory, the pressure of light is evanescent, on obstacles of the atomic or molecular size. But this conclusion is contrary not only to the requirements of many astrophysical data, but also to the experimental results of Lebedew.[2] In some recent communications to the M. N. R. A. S., and the Astrophysical Journal, Professor Eddington[3] has developed a very elegant theory on the " Radiative Equilibrium in the interior of stars," and has successfully explained many of the observational results about the evolution of stars discovered by Russell,[4] Hertzsprung and others. The theory of Eddington is based on the assumption of the existence of radiation-pressure on atoms. We may just quote his own words[5] :—

" As there seems to be a rather widespread impression that gases are not subject to radiation-pressure, it may be advisable to state the theory briefly. The pressure is simply a consequence of absorption or scattering. A beam of radiation carries a certain forward-momentum proportional to its intensity; after passing through a sheet of absorbing medium, a weaker beam emerges carrying proportionately

[1] M. N. Saha, Astrophysical Journal, 1919.

[2] Lebedew—Annalen der Physik, Vol. 32, p. 411 (1910).

[3] Eddington—M. N. R. A. S., Vol. 77, p. 28; the Astrophysical Journal, Vol. 46.

[4] Russell—The Nature, Vol. 93.

[5] Eddington—M. N. R. A. S., Vol. 79, p 28, et seq.

less momentum; the difference of incident and emergent momentum is retained by the medium and constitutes the pressure. The medium, in fact, absorbs the momentum of the beams in the same proportion as it absorbs the energy. The calculations of radiation-pressure on small solid particles are simply calculations of absorption and scattering by these particles; it is not possible to apply such methods to atoms and molecules, which absorb by some internal mechanism. But the relation between absorption and pressure is a perfectly general one, depending only on the conservation of momentum."

In the paper mentioned above, I have tried to prove that the existence of radiation-pressure on gaseous atoms follows as an easy deduction from modern theories of emission and absorption. *It has also been suggested that the action of light pressure is selective.* Let us consider in greater detail what is meant by this term. Suppose a continuous spectrum from a bright back-ground passes through a layer of gas. Then the gaseous atoms will be acted upon by only those pulses of light in the continuous spectrum, which the gas is itself capable of emitting and absorbing. If, for example, the gas be composed of Sodium atoms; then only radium energy contained in the spectral regions about the D_1, D_2-lines and sometimes the other lines of the principal series will act upon the Na-atoms. The remaining part of the continuous light will be without action on the Na-atoms (more later on). Regarded from this point of view, the theory may properly be styled *as the theory of Selective Radiation-pressure.* The object of the present paper is to show that this theory taken along with the modern theories of atomic structure, is capable of explaining many problems in solar and stellar physics, particularly the problems of the radiative equilibrium of the solar atmosphere.

The range of phenomena covered by the works of Eddington is entirely different from ours. Eddington has considered the aggregate effect of light pressure in the interior of stars, *i.e.*, the region where the gaseous atoms are under such a high pressure that they no longer emit, or absorb waves of a particular type, but waves of all lengths. The mass of gas behaves very much like a continuous body, and the radiation pressure is just the same as that given by the continuous theory, for only the aggregate effect is considered. But the class of phenomena which will be discussed in this article refers to the atmospheres of luminous bodies, where the pressure is so low that the gaseous atoms are capable of emitting their own characteristic radiation. The general problem of radiative equilibrium has

already been discussed by Schwarzschild.[1] Before taking up these discussions, I shall give a brief sketch of the problems before us.

§ 2. The Problems of the Solar Atmosphere.

It is well known that the customary division of the sun into the photosphere, the reversing layer, and the chromosphere is based upon the results of spectroscopic observations alone. The correlation of these data to actual physical conditions of temperature, pressure, and distribution of mass is a rather tough problem, and one may find in this connection views which are poles asunder. When we speak of the photosphere, we tacitly assume it to be a sharply defined body like a piece of white-hot iron. The reversing layer and the chromospheres are assumed to be similar to the lower and the upper layers of our own atmosphere. In the discussion which follows, we stick to the view that the photosphere has a sharp, though gaseous boundary, and radiates like a black body at a temperature of 7600°K.[2]

The problems may be briefly grouped under the following headings :—

(1) The enormous distance to which the atmosphere extends over the photospheric disc.

(2) The anomalous distribution of elements in the solar atmosphere.

(3) The radiating power of the different parts of the solar disc.

(4) Unsteady phenomena, *viz.*, Spots and prominences.

The main points of the first problem are very well-known. The value of the gravitational acceleration on the disc of the sun is 27·7 times the value of gravity on the earth, while the temperature is nearly 6000°K. The radial gradient of the density (*i.e.*, rate of decrement of mass per unit volume with height) should therefore be very large, no matter in whatsoever way the temperature may vary in the atmosphere. Let us consider in succession, the three theories of equilibrium.[3]

[1] Schwarzschild—Gött Nachrichten 1906, p. 41.

[2] Felix Biscoe—Astrophysical Journal, Vol. 46, p. 355 (1917).

[3] Most of this discussion is taken from Schwarzschild's paper referred to above, and a paper by D. Brunt, M.N.R.A.S., Vol. 72.

(1) The Isothermal Equilibrium :—Temperature is supposed to be uniform throughout the atmosphere and equal to 6000°k.

Let N_o = number of atoms of a certain element per unit volume just over the photospheric disc, and N = corresponding number at a height z. Then

$$\ln\left(\frac{N}{N_o}\right) = \frac{Mgz}{k\theta} = \frac{mgz}{R\theta}. \qquad 1)$$

Where R = gas constant = 8.30×10^7, k = Boltzmann's gas constant, M = weight of an atom, m = weight of a gram-atom, θ = Absolute temperature.

Taking $\theta = 6000°$K, $g = 27{\cdot}7 \times 981$ *cm.* the logarithmic decrement is $\frac{27.7}{6000} / \frac{1}{300} = \frac{27}{20}$ times the value of the corresponding quantity on the earth. In the case of the Hydrogen atom the density reduces to $\frac{1}{10}$ of its value for a height of 393 km.; for Calcium the corresponding height is only 10 kms. At a height of 1000 kms., the density of Calcium $\rho = \rho_o \left(10^{-100}\right)$. ρ_o = density on the photosphere, *i.e.*, there will be found scarcely one molecule in the whole volume over the disc.

Generally $\rho = \rho_o \times 10^{-\frac{mz}{393}}$ m = atomic weight. z = height in kilometres.

(2) Let us suppose that the temperature does not remain constant, but vary according to the law of adiabatic compression and rarefaction. This will be the case when the atmosphere is the seat of very violent, and turbulent motion, as is the case, to a smaller extent, in the lower atmosphere of the earth (the troposphere). Probably for the lower reversing layer, and the photosphere, the adiabatic law holds good.

The equation of state is

$$\frac{P}{P_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma} = \left(\frac{\theta}{\theta_o}\right)^{\frac{\gamma}{\gamma-1}} \qquad 2)$$

γ = ratio between the specific heats ; from (1), we have

$$\theta - \theta_o = \frac{\gamma - 1}{\gamma}\,\frac{Mg}{k}\,(z - z_o)$$

Taking $\gamma=1.66$, the temperature should fall by 1°C. for every 70 metres of Hydrogen. The atmosphere has a definite limit ($t=p=\rho=0$). This limit is 420 *km.* for Hydrogen and 10.5 for Calcium.

(3) The adiabatic equilibrium can take place only in the regions of violent motion. In the upper regions, convection currents are almost absent (except for such occasional outbursts known as the prominences), and the exchange of heat can take place only by radiation.

The theory of radiative equilibrium is due to Schwarzschild. It is based upon Kirchoff's laws of emission and absorption, and primarily deals with the problem of variation of temperature with height and the darkening of the solar disc towards the edge. It is noteworthy that Schwarzschild does not attempt to account for either the great extension of the solar atmosphere or the anomalous distribution of elements. This is due to the fact that following the old continuous theory he regards the atoms and molecules as infinitely small fragments of black body, and finds the radiation pressure to be evanescent on them.[1]

The arguments of Schwarzschild may be greatly simplified by following a method due to Fabry.[2] Fabry has shown that at any point of free space, traversed by radiation, the motion of temperature has no meaning in itself. Bodies having different physical properties will rise to different temperatures, varying within very wide limits.[3]

Suppose we have a spherical black body at a height 'z' over the photospheric disc, and let us suppose that the solar atmosphere has been somehow lifted up. Equilibrium will be established when the heat radiated by the body will be equivalent to the heat received. Let θ_0 = Temperature of the disc, θ = temperature of the body. Then according to Stefan's law we have

$$\theta=\theta_0\left(\frac{\Omega}{4\pi}\right)^{\frac{1}{4}}$$

where Ω = solid angle subtended by the body at the sun

$=2\pi\left(1-\frac{2z}{r}\right)$ approximately

[1] Schwarzschild : Münchener Berichte—1909.

[2] Fabry ; The Astrophysical Journal, Vol. 47.

[3] The maximum temperature must, however, be less than the temperature of the radiating body.

when z is very small. The value of z being generally very small in comparison to r, the radius (the maximum value of $\frac{z}{r}=\frac{14000}{7\times10^{8}}=\frac{1}{50}$ in the case of the H—K lines), the above assumption is quite justified.

Let us now see how far the actual conditions in the sun differ from these assumptions. F. Biscoe[1] has recently discussed the vast amount of data collected by the Smithsonian Astrophysical Laboratory on the distribution of intensity for different wave-lengths from different parts of the solar disc. He finds that the photosphere radiates like a black body at a temperature of 7500°K. The discussion of course, refers to those parts of the solar spectrum which contain no strong absorption lines. To account for the Fraunhofer-lines, we have to assume that the photosphere is bounded by concentric spherical layers of radient gas, the temperature gradually decreasing with height. These gases pick out and absorb from the continuous spectrum those pulses which they themselves are capable of emitting, so that these regions of absorption appear relatively dark. The intensity of the dark regions corresponds to the intensity of the outermost layers of the emitting and absorbing gas.

If we suppose that a small spherical black body is placed at a point within the solar atmosphere, the radiation received by it is composed of (1) the radiation from the photosphere attenuated by scattering and general absorption ; (2) radiation from the radiant gas of the solar atmosphere interior to the body ; (3) radiation from the radiant gas exterior to the body.

Schwarzschild[2] finds that the combined radiation from the first two causes may be put equivalent to $\sigma\theta'^{4}_{o}$ where θ_{o}' is called the effective temperature. The radiation from the layers exterior to the body

$=\int_{-\infty} E\rho\epsilon\, dz$, where E=volume-emission per unit mass, ϵ=absorption per unit mass.

The integral is equal to

$$=\sigma\theta'^{4}_{o}\ \frac{\epsilon}{g}\int_{-}^{z}\rho g\,dz=\sigma\theta'^{4}_{o}\ \frac{p}{g}$$

Hence $2\sigma\theta^{4}=\sigma\theta'^{4}_{o}\left[1+\frac{\epsilon}{g}p\right]$, σ=Stefan-constant.

[1] F. Biscoe—Astro. Journal, Vol. 46 (1917).

[2] Schwarzschild—Loc. cit.

$$\theta = \frac{\theta'_0}{2^{\frac{1}{4}}}\left(1+\frac{\epsilon}{g}p\right)^{\frac{1}{4}} = \tau\left(1+\frac{\epsilon}{g}p\right)^{\frac{1}{4}}$$

These assumptions and calculations are very rough. With the aid of this relation between temperature, and pressure, we can easily calculate the density. We have

$$z = \frac{k}{Mg}\int\frac{4\theta^4 d\theta}{\theta^4-\tau^4} = \frac{k}{Mg}\left[4\frac{\theta}{\tau} + \log\frac{\theta-\tau}{\theta+\tau} - 2\tan^{-1}\frac{\theta}{\tau} + \text{const}\right],$$

$$\text{and } \rho = \frac{p}{R\theta} = \frac{g}{R\epsilon}\left(\frac{\theta^4-\tau^4}{\theta\tau^4}\right)$$

Brunt[1] has calculated the variation of temperature and density with height on the basis of the above formulæ; the figures are reproduced below :—

z in *km.*	z in angular measure	θ	$\theta-\tau$	ρ.
∞	∞	$5050°K=\tau$	0	0
$2{\cdot}1\times10^5$	$5'$	,,	0	10^{-1636}
$4{\cdot}2\times10^4$	$1'$	,,	0	10^{-3331}
$3{\cdot}5\times10^4$	$\cdot5''$	,,	,,	10^{-34}
$3{\cdot}97\times10^2$	$\cdot5''$	5051	1	3×10^{-6}
$2{\cdot}75\times10^2$	$\cdot4''$	5060	10	3×10^{-5}
0		6000°	950	3×10^{-3}.

This table shows that Schwarzschild's theory leads to an incomprehensible atmosphere with uniform temperature. At a height of 3500 km., where according to the evidence of flash-spectra, lines of H_2, *Ca*, *Fe*...are quite plentiful, we obtain a density of 10^{-34}, or only one molecule of Hydrogen in 10^{15} c.c. of the gas. Schwarzschild's theory therefore fails to account for the great extension of the solar atmosphere.

For a more rigorous application of these formulæ we require more precise information about the pressure, and rate of variation of pressure in the photospheric level, and in the reversing layer, as well as the distribution of elements in these levels (*i.e.*, number of atoms of a particular kind per unit volume). The level attained by an element is generally obtained from the length of the arc of the characteristic line of the element in the flash spectrum.[2] Now this level is generally different for different lines (*vide* Sec. 4) of the same element. Apart from this difficulty, we have to re-

[1] Monthly Notices, R. A. S., Vol. 73.

[2] Mitchell—The Astrophysical Journal, Vol. 38, December.

member that no conclusion is possible about the minimum radiation density[1] of an element from the chromospheric level alone, unless our knowledge is supplemented by auxiliary laboratory experiments. These points will be further considered in the next section. But from what has been said, it is quite clear that none of the three theories sketched above can account for the observed extension of the solar atmosphere.

It seems to be the general opinion of the astrophysicists that there is some sort of repulsive force on the sun which neutralizes the greater part of gravity. It is also supposed that the prominences, particularly, the eruptive ones, are due to some cause which enables "this force of levity" to overcome largely, and in the case of eruptive ones, to preponderate over the force of gravitational attraction on the Sun. We quote the opinions of two distinguished astrophysicists in this connection.

"The rising prominences in some cases observed at Kodaikanal move with an accelerating velocity to be driven entirely away into space by a force opposed to gravity."

(Evershed, Astrophysical Journal, Vol. 28, p. 79.)

Professor Julius, whose views about the interpretation of astrophysical phenomena are so radically different from those of Evershed, writes in a similar strain :—(Astrophysical Journal, Vol. 38, p. 132).

"From the astrophysical point of view, one of the questions material to the explanation of solar phenomena is: What can be presumed about the *general radial gradient of the density in the layers* we are concerned with ?

The subject has been treated very fully and ingeniously on the basis of thermodynamics by Emden in his book 'The Gas-Kugeln'— Emden arrives at the conclusion already mentioned above that the fall of the density must be extremely rapid, but the inference is open to doubt, for in his calculations, Emden presupposes gravitation to be the only radial force acting on solar matter. According to the present state of our physical knowledge, however, we decidedly must admit that on *the sun, gravitation is counteracted by the pressure of radiation*, and by the emission of electrons, and perhaps of other charged particles.

[1] Minimum radiation-density—The minimum number of radiating particles per unit volume which is required or affecting the photographic plate within a certain interval.

Basing on purely theoretical grounds an estimate of the intensity of that counteraction would, for the present, be as rash as denying its existence; but some evidence in favour of its essentiality is given by the fact that many solar phenomena are much better understood if we assume a radial gradient many times smaller than the one that would correspond to gravitational conditions only. In this connection we call attention to the puzzling properties of quiescent, hovering prominences. Father Fenyi, in his interesting discussion of the long series of prominence observations made at the Haynald Observatory, Kalosca, is very positive in his assertion that several well-established facts concerning quite prominences can be accounted for only if in the solar atmosphere gravity is reduced by *certain repulsive forces to a small fraction (something of the order of 1/80)* of its commonly accepted value."

In the papers[1] already referred to, Eddington has calculated the compensation of gravity due to radiation-pressure in the interior of a star of given mass, average density, and luminosity. If G denotes the acceleration due to gravitational attraction alone, and γG the repulsive acceleration due to radiation-pressure, the effective value of gravity reduces to $(1-\gamma)G$. For a star of the size and mass of the sun, Eddington calculates that $\gamma = \cdot 106$, for a molecule of weight 2, and $\cdot 943$ for a molecule of weight 54, so that the effective values of gravity in the interior are $\cdot 894G$ for H_2, and $\cdot 057G$ for a molecule of weight 54. But as we shall see later on, these calculations apply only to the interior of the star. In the atmosphere, quite a different procedure is to be adopted.

The origin of the "force of levity" has been looked for in two other directions excepting radiation pressure—*viz.*, (1) the existence of electrical forces, (2) diminution of gravitational attraction with temperature. There is not much theoretical or experimental investigation to support the second case, while the first case is rather obscure and problematic. Radiation-pressure has been so long at a 'discount' because relying upon the deductions of the continuous theory, we had to admit that it was evanescent on particles of the atomic size. But if the views presented in my paper already referred to be found acceptable, this objection can no longer be held as valid.

[1] Eddington. M. N. R. A. S., Vol. 77, pp. 16 and 596.

The Anomalous Distribution of Elements in the Solar Atmosphere.

The problem *of finding out the distribution of elements in an atmosphere enveloping a dark planet like our earth* can be easily attacked with the aid of the kinetic theory of gases, and both theory, and observations (as far as they go) teach us that the lightest elements are found highest of all in the atmosphere. For the exact calculation of the proportion of elements at any height, we require certain initial data—the number per unit volume of the molecules of a certain element on the surface. The following table taken from Jeans's Dynamical Theory of Gases shows the proportion of different gases at different heights of our atmosphere.

Gas.	Molecular weight.	Number of molecules per c.c. at a height z in kilometres.			
		$z = 0$	$z = 20$	$z = 80$	$z = 160$
Hydrogen	... 2	10×10^{13}	8×10^{13}	430×10^{11}	182×10^{11}
Helium	... 4	1×10^{13}	$2{\cdot}6 \times 10^{13}$	73×10^{11}	13×10^{11}
Neon	... 20	$12{\cdot}5 \times 10^{13}$	$1{\cdot}4 \times 10^{13}$	$\cdot 3 \times 10^{11}$	5×10^{7}
Nitrogen	... 28	780600×10^{13}	42900×10^{13}	520×10^{11}	35×10^{7}
Oxygen	... 32	210000×10^{13}	7000×10^{13}	25×10^{11}	$\cdot 3 \times 10^{7}$
Argon	... 40	9400×10^{13}	139×10^{13}	$\cdot 04 \times 10^{11}$	10^{2}
Krypton	... 83	$\cdot 5 \times 10^{13}$	10×10^{9}	0	...
Xenon	... 130	$\cdot 06 \times 10^{13}$	10×10^{6}	0	...

The table shows that though the amount of Hydrogen on the surface of the earth is very small compared with the proportion of Oxygen and Nitrogen, at a height of 160 km. the whole atmosphere is entirely composed of Hydrogen. This deduction from theory seems to be borne out from observations of meteoric flashes in the higher parts

of our atmosphere. If the proportion of Hydrogen on the surface were greater, the pure-Hydrogen atmosphere would have been reached at a much lower level.

But, as is well-known, spectroscopic observations of the solar chromosphere tell a quite different tale. The lines which are found highest of all are not the Hydrogen lines $H\alpha$ and $H\beta$, but the Calcium lines H and K (highest level=14000 km.). This fact has long remained a puzzle in solar physics. When the fact was first discovered by Huggins, he at first refused to believe that an element like Calcium with an atomic weight 40 times that of Hydrogen would exceed the latter in height, and that by such a large amount. Huggins at first ascribed the H-K lines to some subtle form of Hydrogen, but laboratory experiments have repeatedly shown that the H-K lines belong to no other element except Calcium.

While the puzzle still remains unexplained, later eclipse, and prominence observations have added more puzzling problems of a similar nature.

It appears that the Calcium atoms emitting the H and K lines are not the solitary exceptions to the general rule—"*the lightest elements should be found highest of all.*" Many strong lines[1] of Strontium, Iron, Magnesium, Titanium rival the hydrogen lines in the height reached, though as a general rule, the lines of heavy elements occur in the lower layers. Recently the problem of finding out the level of a line has been attacked by St. John[2] from a quite different standpoint, *viz.*, —variation of the Evershed effect with the intensity of the Fraunhofer lines. St. John[3] finds as a general rule that the stronger Fraunhofer lines show the largest radial motion in spots, and occur at the highest levels,—a result which is in very good accordance with the results of investigation of the flash spectrum during total solar eclipses. Commenting upon the anomalous behaviour of the Calcium and other lines, St. John remarks[4]—

"The high level of Calcium as shown by the H-K lines stands out as strikingly on the chart (which is reproduced below) as in eclipse spectra and remains still an enigma."

[1] Sometimes known as Lockyer's Enhanced Lines.

[2] St. John, The Astrophysical Journal, Vol. 37, p. 322, Vol. 38, p. 341.

[3] St. John, The Astrophysical Journal, Vol. 38, p. 346.

[4] St. John, The Astrophysical Journal, Vol. 37, p. 342.

In spite of many attempts to find a suitable physical explanation of these phenomena, we seem to be no nearer to the real solution of the problem than when it was first detected by Huggins. It is well-known that according to Julius, the whole chromospheric phenomena including the flash spectrum, the spots and prominences are optical illusions due to the anomalous dispersion of the photospheric light by the gases of the solar atmosphere. But the theory has been shown to be quite untenable.[1]

According to Du Gramont,[2] the high level chromospheric lines are "the raies ultimes" of the elements—*i.e.*, they are the last lines to remain when the amount of the element is gradually diminished in the arc or the vacuum tube. In other words, the smallest amount of Calcium, or Strontium is sufficient to show the H-K lines or the Sr. lines 4215, 4077. But as we have seen already, the radial gradient of density for uncompensated gravity over the disc is $\frac{1}{10}$ per 400 kms. of Hydrogen, and $\frac{1}{100}$ per 400 kms. for Calcium. At a height of 8000 kms., the density of H_2 will amount to (10^{-20}) for H_2, and (10^{-800}) for Calcium of their corresponding values on the photospheric level. Supposing 10^{19} molecules of Hydrogen are present per unit volume near the disc (this estimate is based on the pressure measurements of Evershed and Buisson and Fabry[3] in the reversing layer) there will scarcely remain 1 molecule of H in one c.c. at this height, and 1 molecule of Calcium in the whole volume over the disc at this height.

Du Gramont's experiments thus fail to afford the real key to the solution of the riddle.

On the other hand, experiments of a similar nature, if properly conducted, may afford us valuable information about the minimum radiation density of an element in order that a certain line may just be seen under a given stimulus. Without some information on this point, it is not possible to interpret chromospherical phenomena in a quantitative manner. At present, experiments on a quantitative basis are totally lacking. We can only make the roughest calculations with the aid of Schwarzschild's empirical rule on the blackening of photo-

[1] See Numerous Papers in the Astrophysical Journal by Hale, St. John, Adams, Julius, etc.

[2] Du Gramont, Comptes Rendus, Vol. 145.

[3] Buisson and Fabry, The Astrophysical Journal, Vol. 31.

graphic plates, and works like those of Wood, Bevan, Ladenburg, and St. Loria on the number of radiant centres in a given mass of glowing gas.

§ 4.

For a more systematic study of the phenomena, I add the following tables compiled from Mitchell's list of Chromospheric lines. (Astrophysical Journal, Vol. 38, p. 407).

The characteristic lines of each element have been tabulated in separate tables, and the height shown against the wave-length of the lines. The lines have been arranged according to their sequence in the series. In the case of Helium, 6 tables are given, corresponding to the six known series of the element.

The remaining metals treated here, Mg, Ca, Sr, Ba, show complicated types of spectra which have been fully discussed by Saunders, Fowler, and Lorenser.[1] In the present classification, I have followed the most up-to-date results which may be stated here. The series spectra of Alkaline earth metals can be grouped under three headings :—

(*i*) Series composed of single lines; we shall denote them by the subscript 1. Thus Ca I-series will mean the Calcium series of single lines.

(*ii*) Series composed of double lines. Ca II-series means the group of Calcium-series of double lines.

(*iii*) Series of triplet lines. Ca III-series means the group of Calcium series of triplet lines.

Now the group of series contained under each one of these headings consist of the Principal series, the Sharp series, the Diffuse series, and the Fundamental series. Thus the symbol Ca I. P-series—denote the Principal series of single lines of Calcium. Ca III. F-series—mean the Fundamental triplet series of Calcium. The series formula for each line has been added. The notation used is that of Konen.[2]

[1] Saunders, Astro. Journal, Vol 29. Fowler, Phil. Trans, Vol. 214.

[2] Leuchten der Gase und Dämpfe-Kapitel 1, II, Teil 2.

LEVEL ATTAINED BY THE BALMER-SERIES OF H-Lines in the SOLAR CHROMOSPHERE.

(Taken from Mitchell-Astrophysical Journal, Vol. 38, 1913).

Line	Wave-length in Angstroms.	Height in Kilometres.	Line	Wave-length in Angstroms.	Height in Kilometres.
1 H-$_\alpha$	6563		19 H$_\tau$	3679·00	2000
2 H-$_\beta$	4861·90	8000	20 H$_\upsilon$	76·151	1500
3 H$_\gamma$	4340·63	8000	21 H$_\phi$	73·91	1500
4 H$_\delta$	4102·00	8000	22 H$_\chi$	71·62	1500
5 H$_\epsilon$	3970·18	8500	23 H$_\psi$	69·61	1500
6 H$_\zeta$	3889·20	8500	24 H$_\omega$	67·40	1500
7 H$_\eta$	35·53	7000	25	66·24	1500
8 H$_\theta$	3798·05	6000	26	64·82	1500
9 H$_\nu$	70·78	6000	27	63·56	750
10 H$_\kappa$	50·30	6000	28	62·40	750
11 H$_\lambda$	34·51	6000	29	61·38	500
12 H$_\mu$	22·08	6000	30	60·42	500
13 H$_\upsilon$	12·12	6000	31	...	...
14 H$_\xi$	04·00	4000	32	58·78	400
15 H$_o$	3697·35	3000	33	58·07	400
16 H$_\pi$	91·70	2500	34	57·41	400
17 H$_\rho$	86·98	2000	35	3656·80	400
18 H$_\sigma$	82·95	2000	Limit	3613·14.	

We next take the Helium-lines of which 15 members have been unambiguously identified by Mitchell in the range investigated (λ=6191·6-to λ=3318). It is well-known that ordinarily the Helium-lines can be arranged into 6-groups, the P-series, the S-series, and the D-series of He I, and the corresponding series of He II. (Doublets) We have accordingly arranged the lines in 6-series and shown the heights against the line. In certain cases, the line may be too near a strong line of another element for separate identification *e.g.*, the line 3888.98 which is the second line of the P-series of He I, is masked by H_ζ. In other cases, it is not possible to ascribe the lines to He unambiguously; such lines have been marked by (?). I am unable to account for the absence of the strong line 5048 from Mitchell's list; probably some error is involved here. Many strong lines, *e.g.*, 6678, 7281 are present, but are beyond the range explored by Mitchell. It will be noticed that He -II lines occur at a much higher level than the He I lines. This is in accordance with the observations of Runge and Paschen[1] that He II lines are always more prominent than the He I lines which occur only when a rather large quantity of the gas is employed.

The Chromospheric Level reached by Helium-lines.

He I-P-Series.

No.	1	2	3	4	5	6	7	8	9
Length.	20587	5015·7	3964·87	3613·78	3447·731	3354·66	3296·9	3258·3	
Height.		1600	1000	750 ?	300 ?	300 ?			
Remarks.	beyond the range.			Blend O	Blend Zr-Fe	Blend Ti	beyond the range.		

[1] Kayser :—Handbuch der Spektroskopie, Bd. V, Helium.

He I-D-Series.

No.	1	2	3	4	5	6	7	8	9
Length.	6678	4922·10	4388·04	4144·05	4000 42	3926·68	3871·95	3833·71	3805·90
Height.		1500	2000	1000	1000	600 ?	500 ?	500 ?	
Remarks.	Outside range.					Blend Fe.	Blend Fe	Blend C	

He I-S-Series.

No.	1	2	3	4	5	6	7	8	9
Length.	7281·81	5047·86	4437·72	4169·31	4024·13	3936·061	3878·33	3838·24	
Height.		?	750	1500	500 ? 750 ?	750 ?	?	?	
Remarks.		Absence inexplicable.			Blend Ti Zr Fe				

He II-P-Series.

No.	1	2	3
Length.	10834·4	3888·78	3187
Height.		8500?	
Remarks.	Beyond range.	Masked by H_ζ	These are all outside the range.

He II-D-Series.

No.	1	2	3	4	5	6	7	8	9
Length.	5875·87	4471·7	4026·5	3820	3705	3634	3587	3555	
Height.	7500	7500	6000	6000	750	400	400 ?	350 ?	
Remarks.							Masked by Al-Fe	Blend Fe	

He II-S-Series.

No.	1	2	3	4	5	6	7	8	9
Length.	7065·48	4713·25	4120·97	3867·6	3733·004	3652·12	3599·47	3563·12	
Height.		6000	1000	750	350 ?	?	?	?	
Remarks.	Beyond range.			Blend Fe-C.	Blend V.				

According to the theoretical investigation of Bohr,[1] (as also supported by the experimental work of Fowler),[2] the lines formerly ascribed to the hypothetical element cosmic hydrogen (the lines of the Pickering series, and the Rydberg series) have been shown to be due to *ionised Helium i.e., Helium which has lost one electron.* It is a very remarkable fact that Mitchell has indentified the line $\lambda=4686$,

$$\upsilon=N\left[\frac{1}{(1+\cdot5)^2}-\frac{1}{2_2}\right],$$ in the Chromosphere, and the level reached by it is 2000km. None of the lines of the Pickering series

$$\upsilon=N\left[\frac{1}{2^2}-\frac{1}{(2\cdot5)^2}\right]$$ are however found. This shows that there may be probably some mild type of ionisation in the lower layers of the solar atmosphere.

[1] Phil. Mag, July 1913. [2] Fowler, M. N. R. A. S. 1912.

The Ca-lines. (Atomic weight 40).

The total number of Calcium lines on Mitchell's plates is 35. The following table shows the classification of the lines according to series, and the height reached by each line.

Ca I.

Line.	Height reached in km.	Symbolic Formula.	Remarks.
4226·91 (g)	5000	$S^{I}(0)-P^{I}(1)$	Other members are beyond the range.

Ca II.

Line.	Height reached in km.	Symbolic Formula.	Remarks.
3968·63 (H)	14,000	$S^{II}(0)-P^{II}_{I}(1)$	Other members beyond the range.
3933·83 (K)	14,000	$S^{II}(0)-P^{II}_{II}(1)$	
3706·18	750	$P^{II}_{I}(1)-S^{II}(1)$	”
3737·08	1500	$P^{II}_{II}(1)-S^{II}(1)$	

Ca III.

Line.	Height reached in km.	Symbolic Formula.	Remarks.
4425·61	600	$P^{III}_{I}(1)-D^{III}_{I}(2)$	
35·13	600	$P^{III}_{II}(1)-D^{III}_{II}(2)$	Diffuse series 1st members.
35·86	600	$-D^{III}_{I}(2)$	
4454·97	500	$P^{III}_{III}(1)-D^{III}_{III}(2)$	”
56·08	500	$-D^{III}_{II}(2)$	
56·81	350	$-D^{III}_{I}(2)$	

3624·15	400	$P_{I}(1)-D(3)$	Separation indistinguishable.[1]
30·82	?	$P_{II}(1)-D(3)$	
44·45	400	$P_{III}(1)-D(3)$	
6102·99	300	$P_{III}(1)-S(1)$	The Sharp series.
3949·09	?	$P_{I}(1)-S(2)$	
57·23	400	$P_{II}(1)-S(2)$	„
73·83	750	$P_{III}(1)-S(2)$	
3468·68	?	$P_{I}(1)-S(3)$	
74·90	?	$P_{II}(1)-S(3)$	„
87·76	400	$P_{III}(1)-S(3)$	
4586·08	400	$D_{III}(1)-F(1)$	The fundamental or the Bergmann series.
81·59	350	$D_{II}(1)-F(1)$	
78·71	350	$D_{I}(1)-F(1)$	
4098·82	500 (b. Gd)	$D_{III}(1)-F(2)$	
95·09	400	$D_{II}(1)-F(2)$	„
92·93	500 (b. V)	$D_{I}(1)-F(2)$	

The following lines remain un-classified.[2]

5857·72	400	(b-Fe)
5603·14	500	(b-Ce)
5601·50	500	(b-Fe)
5598·67	500	(b-Fe)

[1] The upper subscript III is omitted henceforth.

[2] b. means blend with a line of another element.

5594·73	500	
5590·34	400	
5588·94	750	
5582·20	600	(b Y)
5513·20	350	(Fe ?)
5270·49	500	(Ti-Cr ?)
65·85	350	(Ca-Cr ?)
64·37	350	(Ti ?)
62·39	300	(Cr ?)
61·88	300	(Fe ?)
5041·89	500	(Ti-V ?)
4318·91	600	
4307·91	750	
4302·89	750	
4299·58	550	
4289·52	1300	
4283·17	700	

The Sr-lines. (Atomic weight—87·63).

The total number, unambiguously identified, is 12.

Sr I.

Line.	Height in km	Remarks.
4607	350	The second line is at 2932. The small height reached by this line is rather difficult to account for.

Sr II.

4215·66	6000	First members of the Principal pair-series.
4077·88	6000	
4161·95	600	Sharp-Pair series, first member.
4305·60	600	
3464·68	300	Sharp-pair series, second members.
3475·01	300	

Sr III.

None of the series of Sr III seem to be present, though the first lines of many of them are within the range. The following lines remain unclassified.

5543·41	350
5540·30	300
5535·06	600
5451·27	300
5330·18	350
5257·00	300
4438·22	350

The Ba-lines. (Atomic weight 137·37).

The total number in Mitchell's list is 11. There are, besides, many lines which may be due to Barium.

Ba I.

Line.	Height in km.	Remarks.
5535·69	400	The second line is 3275, and is beyond the range.

Ba II.

4934·24	750	The first members of the pair P-series.
4554·21	1200	
4225·19	350	Sharp pair series. The second line is a blend with an iron line
4900·13	600	

Ba III.

The diffuse series, do not seem to be present, except the doubtful lines like 5535·69. Similarly, the sharp series do not seem to be present except a few doubtful lines. The fundamental series, would be far down in the infra-red. The following lines remain unclassified.

6141·93	500	)
5997·6	300	)
5853·91	400	)

The Mg.-lines.

(At W = 24·32).

Total Number = 10.

Mg. I, as well as Mg. II, lines seem to be entirely absent, or beyond the range. The first line of the P-series of Mg. I is 2852.

Mg. III.

Line.	Height in km.	Remarks.
5167·55	700	S-Series, the b-group.
72·87	1000	
83·88	1200	
3829·51	6000	D-Series.
32·46	6000	
38·44	7000	

The following lines remain unclassified:—

5528·91	400	According to Saunders, these lines are the 3rd, 4th and 5th members of a series of which the preceding members are 1·7108μ; ·8807μ.
4703·16	500	
4352·02	600	
4571·25	400	
4481·39	400	

From a scrutiny of the table we can make the following deductions.

(1) Generally, the first lines of a series reach the highest level. The lower is the position of the line in a series, the lower is the level reached by it.

(C/o—the table of H-lines, and He lines).

It is unfortunate that most of the lower lines of the series spectrum of alkaline earths are too far away from the region explored by Mitchell to serve as a test of the present hypothesis.

(2) In certain cases, the second or the third line may exceed the level reached by the first line, but this invariably occurs when these latter lines are very near the spectral region ($\lambda=3900$), while the first line is at greater distance from this region. Compare, for example, the Hydrogen lines $H\epsilon=3970$, $H\zeta=3889$, which reach a level of 8500 km, while their predecessors $H\beta$, $H\gamma$, H , reach the level of 8000 km. Similarly consider the HeII-D-series. The second line 4472 reaches the same level as the first line 5876, (7500 km), while the third and fourth lines do not lag much behind (6000 km).

(3) The lines of the Principal and the Diffuse series are always better represented than the lines forming the Sharp-series.

We shall mention here another important remark of St. John. " It will be noticed that the heights do not follow the effective sensitiveness, that in the region $\lambda=3700$ to $\lambda=4350$ Å.U, the heights are much greater than in the region $\lambda=4350$ to $\lambda=5000$, though the sensitiveness is greater in the latter case, and that in the region 5000 to 5800, the heights are not low because of the low sensitiveness of the apparatus, which for this region averages higher than at $\lambda=3850$."

These remarks are quite in agreement with the remarks made in (2) above.

I have not tried to classify the Titanium, Chromium, Scandium, Vanadium, and Aluminium, Iron and Nickel lines, because though the lines are very plentiful in the Chromospheric spectrum, no series formulæ are known for them.

The above analysis shows that if, " radiation pressure " be the cause of the " reduction of gravity " on the surface of the sun, *it acts very differently on atoms of different elements.* The normal force on an atom of Calcium is 40 times the force on an atom of Hydrogen, but the mere fact that Calcium is found higher than Hydrogen, shows that the gravitational force on Calcium is so strongly neutralized that it falls considerably below the value for Hydrogen. Similarly the compensation for Strontium, Barium, Magnesium, and other high-level Chromospheric elements is very large. We have further to account for the " falling in level " of the more refrangible lines of the series. In the following section we shall see how far the theory of selective radiation pressure is capable of coping with these difficulties. But for this purpose, we must begin with a preliminary discussion of modern theories of atomic structure and atomic radiation.

§5

Radiation from Atoms and Atomic constitution.

The classical researches of Rydberg, Ritz and others have established the fact that the frequency of the characteristic lines of an element forming a series can be expressed by the formula,

$$\nu=N\left[\frac{1}{[f(m)]^2}-\frac{1}{[\psi(n)]^2}\right]$$

where m and n are pure integral numbers, and

$$f(m)=m+a+\frac{\beta}{m}+\frac{\gamma}{m^2}+\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$\psi(n)=n+a'+\frac{\beta'}{n}+\frac{\gamma'}{n^2}+\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

The simplest illustration is the Balmer series of Hydrogen lines in which $a=\beta=a'=\beta'=\quad\ldots\quad\ldots\quad\ldots=o$

$$m=2.\quad n=3,\ 4,\ 5 \text{ etc}\quad\ldots$$

Bohr has explained Balmer's law in terms ef Planck's quantum theory, and Rutherford's model of atomic constitution. For our purpose, we can summarize the main features of the theory in the following manner for Hydrogen.

The Hydrogen atom consists of a positive nucleus (charge $+e$, mass M), and a planetary electron (charge e, mass m). The orbit of the electron can be calculated from the classical theory, but it appears that all orbits so obtained do not actually represent a stable configuration. Bohr introduces the hypothesis that only those orbits are stable in which the moment of momentum of the system is some multiple of Planck's constant h divided by 2π.

We can, in this way, obtain a number of stable orbits which are characterised by the quanta numbers 1, 2, 3, m. Bohr has calculated the energy of these orbits. The energy is given by the quantities

$$A-\frac{Nh}{1^2},\quad A-\frac{Nh}{2^2},\quad\ldots\quad\ldots\quad A-\frac{Nh}{m^2}.\quad\ldots$$

Where N = Rydberg constant $=\frac{2\pi^2e^4m}{h}$, and A is some constant.

A pulse of frequency $\nu=N\left[\frac{1}{2^2}-\frac{1}{3^2}\right]$ is emitted when the electron passes from orbit (3) to orbit (2). Conversely in the case of absorption, the Hα line can be absorbed only when the electron passes from the orbit (2) to the orbit (3).

The Balmer-series of lines

$$\nu=N\left[\frac{1}{2^2}-\frac{1}{m^2}\right]$$

are therefore emitted, when the electrons pass from the orbit ($m=3, 4, 5$..............) to the limiting orbit (2), and they are absorbed only when the electron pass from the initial orbit 2 to the orbits (3, 4, 5......m...........).

These investigations make it quite clear that if we are given a certain quantity of Hydrogen gas, the necessary condition that under a certain stimulus it would emit a Balmer line is that the stimulus must produce a sufficient number of Hydrogen atoms in which the electron is in the orbit (3), (4), (5) etc. Conversely in order that a given mass may absorb the Balmer lines, when traversed by a continuous spectrum, it must contain a sufficient number of H-atoms in which the electron is in the orbit (2), and the continuous spectrum must be very rich in pulses of frequency

$$\nu=N\left[\frac{1}{2^2}-\frac{1}{m^2}\right].$$

Now it can be safely assumed that Hydrogen gas at ordinary temperature, and not subjected to any stimulus, electrical or thermal, consists of atoms in which the electron is in the state (1). When such a mass is traversed by a continuous source of light, it can only absorb lines of frequency

$$\nu=N\left[\frac{1}{1^2}-\frac{1}{m^2}\right],$$

a series which lies far up in the ultra-violet and which has been found in the emission-spectrum by Lyman.[1] The ordinary Balmer-lines would not be absorbed.

This accounts for the repeated failures of experiments to obtain reversal of H-Balmer lines and the ultimate success of Ladenburg and St. Loria,[2] who obtained reversal of Hα and Hβ by passing continuous radiation through a glowing vaccum tube containing Hydrogen.

Let us now turn our attention to the conditions in the sun. It has been remarked that Hydrogen at ordinary temperatures is com-

[1] Lyman, The Astrophysical Journal, Vol. 43.

[2] Ladenburg and St. Loria;—Ann. d. Physik. Vol. 38, 1912.

posed only of the atoms with the orbit (1), but it is quite possible that if the temperature be sufficiently raised, a large number of atoms with the orbit (2) may be produced as a result of molecular collision, and such a mass will show the Hα absorption when traversed by a strong continuous light. In the sun, we have exactly this state of things,—a long column of H-gas at a very high temperature backed by a photosphere at a teperature of 7600°K.

Radiation and Absorption of other elements.

As a result of the radio-activity, and X-Ray experiments, our knowledge of the constitution of the atom has become much clearer in recent years. It is now believed that the atom consists of a positive nucleus with the net charge of $(+ze)$, where z=atomic number (place of the element in the periodic table), with z negative electrons moving in proper orbits about the nucleus. The application of Bohr's theory to a system like this involves knowledge of the solution of the problem of 3, or z bodies which, as is well-known, has not yet been successfully solved. Bohr[1] has however shown that there are reasons to believe that the ordinary series spectra with the characteristic constant N are due to the vibrations of a single electron,—the nucleus, and the remaining $(z-1)$ electrons behaving as a single charge.

Owing to the formidable mathematical difficulties connected with the solution of the problem of 3 bodies,[2] it has not yet been found possible to extend Bohr's theory even to the case of Helium, which, as we know, has a very simple constitution (charge $+2e$, mass 4 M, number of planetary electrons 2). But the analogy with Hydrogen leads us to believe that Helium has got 6 ultraviolet series which stand to the 6 ordinary series in the same relation as the Lyman series $\nu=N\left[\frac{1}{1^2}-\frac{1}{m^2}\right]$ stand to the Balmer series $\nu=N\left[\frac{1}{2^2}-\frac{1}{m^2}\right]$. This leaves the problem of Helium and Parhelium still unsettled. The absorption bands of ordinary Helium, therefore, do not lie in the region of the first lines of the Principal or Diffuse series, but in the region of the first lines of the corresponding Principal or Diffuse

[1] Bohr, Phil. Mag, July 1913.

[2] *Vide* Jeans—on Speclral series and Atomic Structure, Journal of the Chemical Society, 1919.

series in the ultraviolet. This accounts for the nonreversal of ordinary Helium-lines. Some of these lines have been obtained by Lyman[1] in the extreme ultraviolet, but their classification is still obscure. More recently, Paschen, has reversed the D_3-line by employing a method similar to that of Ladenburg and St. Loria.

The alkali metals have pair-series of Principal, Diffuse, and Sharp lines. The series formula of the Principal Lines may be expressed symbolically as—

$$\nu = S(0) - P_{I}(1), \text{ and } S(0) - P_{II}(1)$$

e.g., The first pair corresponds to the D_1-and D_2-lines of Sodium, in which case

$$P_1(1) = \frac{N}{\left(2 + p_1 + \frac{\pi_1}{2^2}\right)^2}, \quad P_{II}(1) = \frac{N}{\left(2 + p_2 + \frac{\pi_2}{2^2}\right)^2},$$

$$S(0) = \frac{N}{\left[1\cdot5 + s + \frac{\sigma}{(1\cdot5)^2}\right]^2}$$

It is very probable that $A - hS(0)$, (A = a constant) represents the energy of the normal (*i.e.*, not subject to any stimulus) atom of Sodium. When a continuous light passes through Sodium vapour, the atom absorbs the D_1-,D_2-lines, and assumes the configuration in which the energy is $A - P_1(1)\,h$ or $A - P_2(1)\,h$. This accounts for the success of the reversal experiments of Wood and Bevan on Alkali metals. Whether all the atoms or only a fraction of them, are in the configuration in which the energy is $A - hS(0)$ or whether there is a still more fundamental configuration with a different energy-content cannot be decided without further laboratory experiments.

The nature of the spectra becomes much more complicated as we proceed to the next group of elements in the periodic table. But the alkaline earth metals are so well represented in the spectrum of the sun that a discussion of their spectra is an indispensable necessity. We have already given a brief sketch of the grouping of their spectra.

[1] Lyman, Astrophysical Journal, Vol. 43.

The formula for the Principal series of single lines is $\dot{S}(0) - \dot{P}(m)$,

for the P-series of Pair-lines is $S^{II}(0) - \left\{ \begin{array}{l} P^{II}_{I}(m), \\ P^{II}_{II}(m), \end{array} \right.$

for the P-series of Triplets is $S^{III}(0) - \left\{ \begin{array}{l} P^{III}_{I}(m) \\ P^{III}_{II}(m) \\ P^{III}_{III}(m) \end{array} \right.$.

In accordance with the conceptions developed in this section, the terms $A - hS^{I}(0)$, $A - hS^{II}(0)$, $A - hS^{III}(0)$ may be taken to represent the energy of the atom of, say Calcium, in very fundamental states. But there is no laboratory or theoretical investigation which can give us much information about the interconnection between these terms. MacLennan has shown when ordinary Calcium (or any alkaline earth-metal) vapour is bombarded with electrons of proper velocity, the Principal lines of the series of single-lines are obtained. The lines of the doublet and triplet series have not been obtained by this method. Fowler has shown that there are combination-series connecting the singlet and the triplet system, but the pair-series stand apart, and probably belong to the "enhanced" type, *i.e.*, they are probably due to a nucleus behaving like a double charge, and require strong electrical stimulus for their generation. More laboratory experiments are required to ascertain the relation among the Ca I, Ca II and Ca III lines and the proper conditions of their generation. The principal series of the triplets have not been discovered and probably lie beyond the visible range.

From these discussions, we come to the important conclusion that we are no longer justified in regarding all the atoms contained in a glowing mass of gas as alike. Such a mass is composed of numbers of different groups with different energy-contents. To make matters

clear, let us take the case of a vacuum tube containing glowing hydrogen. If n be the total number of particles, then

$$n = n_1 + n_2 + \ldots\ldots\ldots\ldots n_m + \ldots\ldots\ldots\ldots$$

where n_m = number of atoms of which the electron is in the orbit m.[1]

The relative proportion of these groups vary with the stimulus, and offers a rich field for investigation. Generally atoms in state (1) are in the largest number, but with the strength of the stimulus, other groups may appear in larger proportions.

The nearest analogy to the solar spectrum are the experiments of Wood and his co-workers already mentioned, in which light from an arc was passed through a tube containing sodium which could be electrically heated up to 600°C. The resulting spectrum was a continuous one with the D_1 and D_2 lines, and other lines of the Sodium P-series reversed. At low temperatures, the reversal was confined only to the leading members, but with increase of temperature, and quantity of gas in the absorbing column, Wood succeeded in obtaining reversals of not less than 58 members of the series.

Bevan applied Wood's method for obtaining reversal of other Alkali elements, and tried to obtain the number of emission-centres of the lines of the Principal-series from the refractive index, with the aid of Lorentz's well-known formula of dispersion :-

$$\mu^2 - 1 = \Sigma \frac{A\lambda^2}{(\lambda^2 - \lambda_0^2)} \text{ . where } A = \frac{4\pi e^2 n \lambda_0^2}{mc^2} \text{ .}$$

λ_0 = wave-length of absorption band. e = electronic charge. m = mass of the electron, n = number of dispersion centres, c = velocity of light.

In the case of Rubidium, the formula found was

$$\mu - 1 = \left[\frac{\cdot 90\lambda^2}{(\lambda^2 - \lambda_1^2)} + \frac{2\cdot 71\ \lambda^2}{(\lambda^2 - \lambda_1'^2)} \right] + \left[\frac{2\cdot 5 \times 10^{-3}\lambda^2}{(\lambda^2 - \lambda_2^2)} + \frac{7\cdot 5 \times 10^{-3}\lambda^2}{(\lambda^2 - \lambda_2'^2)} \right]$$

$$+ \left[\frac{3\cdot 3 \times 10^{-4}\lambda^2}{(\lambda^2 - \lambda_3^2)} + \frac{10 \times 10^{-4}\lambda^2}{(\lambda^2 - \lambda_3'^2)} \right] + \text{etc.}$$

where (λ_m, λ_m') are the mth members of the Principal-pair series.

1. It must be remembered that these numbers refer to the H-atoms only, not molecules. Ladenburg and St. Loria finds that in their experiments only 1 in 50,000 molecules was dissociated into atoms by the passage of the discharge. The spectrum emitted by the molecules is the well-known Secondary Spectrum.

This shows that under the stimulus (here heat), the number of atoms absorbing the first member is about 300 times the number absorbing the second member. In the case of the third member, the proportion is 1 : 3000.

No experiments seem to have been undertaken to find out how the proportion of absorption centres vary with temperature, and other physical causes.

On
The Lifting Power of Radiation Pressure.

The foregoing review of "Atomic Structure and Spectral Radiation" shows that a large amount of theoretical and experimental work is necessary before the problem of the sun can be satisfactorily handled. The following work is therefore offered only as a "tentative hypothesis," for the solution of the problems sketched above.

Our first assumption is that the force of gravity on the sun is largely neutralized by a repulsive force due to the pressure of radiant energy from the photosphere. Let us consider how this can happen. Consider a Sodium-atom in the atmosphere of the sun and let us suppose that to begin with, the atom was in the natural state, *i.e.*, the state in which the energy of the atomic system was $A - h\,S(0)$ (*vide* § 5). Now as the photospheric light falls upon the atom, it absorbs the pulse having the frequency $\nu = S(0) - P(1)$, and its configuration changes to a state in which the energy is $A - h\,P(1)$. In this process, the atom acquires a forward momentum equal to $\frac{h\nu}{c}$, ($\nu = S(0) - P(1)$), in the direction of the incident light. The atom will remit the light $\nu = S(0) - P(1)$, but now laterally, *i.e.*, not in any specified direction, and will be reconverted to the state with the energy content $A - h\,S(0)$. Another fresh process of absorption and emission will then take place. In this way, the atom will be acted upon by successive kicks of photospheric light, amounting to a total force of $\frac{nh\nu}{c}$, where n = number of pulses absorbed in a normal direction to the disc per unit of time. The loss of momentum due to emission cannot accumulate in a specified direction, as in the process of absorption, for emission takes place in all possible directions according to the law of chance.

Let us now see how this term '$nh\nu$' is connected with the photospheric intensity of the light of frequency ν. Let E_λ = the emissivity in the wave-length $\lambda = \frac{c}{\nu}$ from the photosphere. Then $nh\nu = \epsilon E_\lambda$, where ϵ is a quantity which expresses the stopping-power per atom. The force of levity is proportional to μE_λ, where $\mu = \epsilon/c$.

The radial gradient of density of radiant atoms emitting light of wave-length λ is thus given by the formula

$$ln\left(\frac{n}{n_0}\right) = -\frac{(Mg-\mu\ E_\lambda)z}{k\theta},$$

where n_0 = number at zero-level, n = number at the level z.

We have seen that according to F. Biscoe, the photosphere radiates like a black-body at a temperature of 7500° K. From this we can calculate the value of E_λ for any wave-length by using either the Planck-formula or the approximate Wien-formula :—

$$E_\lambda = \frac{A}{\lambda^5}\ \frac{1}{e^{\frac{c_1}{\lambda\theta}} - 1}\ \text{(Planck)},\quad = \frac{A}{\lambda^5}\ e^{-\frac{c_1}{\lambda\theta}}\ \text{(Wien)}.$$

The wave of maximum emission is obtained from Wien's law,

$$\lambda_m\,\theta = b = \cdot 294\ ^{1}$$

$$\therefore\ \lambda_m = \frac{\cdot 294}{7500} = 3920\ \text{A}^\circ.\ \text{U}.$$

The wave of maximum emission is thus seen to be very close to the great Calcium bands H ($\lambda = 3968$), and K (3933), which are the highest lines in the Chromosphere.

The following Table shows the relative values of E_λ from ·1μ to ·6μ taking E_m (emission at $\lambda = 3920$) = 1.

λ	E_λ	λ	E_λ
·1 μ ...	... $\cdot 32 \times 10^{-4}$	·4077 (Sr.II)	... ·995
·12 μ (Hydrogen ·1216)	... $\cdot 53 \times 10^{-3}$	·4227 μ (Ca.Ig.)	... ·985
·15 μ ...	... ·06	·4215 (Sr.II)	... ·986
·2 μ ...	... ·291	·43 μ ...	... ·980
·25 μ ...	... ·573	·44 ...	... ·966
·29 (MgI.)	... ·785	·4554 (Ba.II)	... ·947
·3 ...	... ·8 8	·4607 (Sr.I)	... ·939
·35 ...	... ·961	·47 ...	... ·925
·36 ...	... ·979	·48 ...	... ·909
·38 ...	... ·993	·4994 (Ba.II)	... ·874
·3920 ...	... 1	·5 ...	... ·872
·3933 (Ca.II—K)	... ·999	·5183 (Mg.)	... ·838
·3968 (Ca.II—H)	... ·998	·53 ...	... ·816
		·5535 (Ba.I)	... ·770
		·6 ...	... ·668
		·65 (Hα)	... ·580

[1] Planck—Wärmestrahlung—Kapitel IV.

The above table shows that E_λ has the largest value in the spectral region $\lambda = \cdot 36\mu$ to $\cdot 43\mu$, on both sides of the wave of maximum emission. The force of levity should therefore be greatest for those radiant atoms which have their emission lines in this region.

We have already mentioned that this fact has already been established by St. John from a statistical discussion of the variation of the Evershed effect in spots. Lines in this region show, as a rule, higher level than the lines in the other spectral regions, and consequently also a greater Evershed effect.

The neutralisation of gravity will be greatest for those elements which have got their *principal absorption lines* (*vide* § 6) in the region about the wave of maximum emission. They will also show the greatest level, for a very small quantity of matter suffices for the emission of these lines. This is exactly the case for all high-level Chromospheric lines, *viz.*, Calcium H and K ($\lambda = 3968$, 3933), Calcium g ($\lambda = 4227$), the Sr pair ($\lambda = 4215$ and 4077), and to a smaller extent the Ba pair ($\lambda = 4934$ and 4554), and the Helium lines. The principal pair lines of Magnesium, and of the alkali metals (with the exception of Sodium D_1, D_2) lie too far away from the region explored.

Let us now see wheth r we can quantitatively compare the value of Mg with that of μE_λ. This requires a knowledge of the radiating and absorbing properties of a gas under a purely thermal stimulus. In this direction, both experiments and theories are totally lacking, unless we except the experiments of King which are purely of a qualitative nature.

But there is some amount of evidence from other directions to the effect that the neutralisation is very considerable. Eddington[1] has found that in the interior of a star of the mass and size of the sun, radiation-pressure reduces the value of gravity to ·894 of its value in the case of an atom with the weight 2, and to ·057 in the case of an atom with the weight 54, so that these behave respectively like atoms of weight 1·788 and 3·078. In the interior, the compensation is entirely due to the gradient of temperature, *i.e.*, difference of temperature at two points of a radial filament and is proportional to $\frac{d}{dx}(\mu T^4)$. But in the present case, the balancing pressure is entirely

[1] Monthly Notices, R. A. S., Eddington, loc. cit.

due to the radiation from the photosphere. The pressure should therefore be much larger than in the case treated by Eddington, though it will, to some extent, depend upon the stopping power of the gas.

Eddington's theory is partly based upon theory, partly upon certain empirical results discovered by Hertzsprung, and Russell in their study of the progression of Giant and Dwarf Stars.[1]

In the present case, we shall adopt a similar procedure. We shall assume that in the case of the high level Chromospheric lines, the compensation is almost complete; and then see whether the results are consistent.

Beginning first with the Calcium H-K lines, which are the highest lines in the Chromosphere (highest level reached =14000 Km), let us calculate the radial gradient of density. Let us suppose that on the photosphere, the number of Ca-atom is 10^{20} per c.c., and at the highest level obtained from Mitchell's plates, the number is (10^{8}) per c.c.

$$\text{Then } ln\left(\frac{n}{n_0}\right)=27\cdot6=\frac{Mg-\mu E_\lambda}{k\theta}\,z, \text{ where } z=14000\text{km},$$

$$\text{or } (Mg-\mu E_\lambda)=\frac{27\cdot6\cdot k\theta}{z}=\frac{27\cdot6\times1\cdot34\times10^{-16}\times7\cdot5\times10^{3}}{1\cdot4\times10^{4}\times10^{5}}$$

$$=1\cdot98\times10^{-20}$$

$$\text{Now } Mg=\frac{40}{60\times10^{22}}\times27\cdot7\times981=1\cdot81\times10^{-18}$$

$$\because \mu E_\lambda=1\cdot81\times10^{-18}-1\cdot98\times10^{-20}=1\cdot79\times10^{-18}$$

$\because$ Mg is only slightly greater than μE_λ. In other words, the compensation is almost complete.

Let us now put $Mg-\mu E_\lambda=M(g-g')$, $g'=\frac{\mu E_\lambda}{M}$; then from the above calculation $\frac{g-g'}{g}=\frac{1\cdot09}{100}=\frac{1}{91}$, and $M(g'-g=\frac{40}{90}$ *i.e.*, Gravity is reduced to $\left(\frac{1}{90}\text{th}\right)$ of its normal value, so that Calcium behaves like an element of weight ·44.

[1] Russell, The Nature, Vol. 93.

For radiant Ca-atoms emitting the g-line 4227, $g'=\dfrac{1{\cdot}77\times10^{-18}}{1{\cdot}81\times10^{-18}}$

and $M(g-g')=\dfrac{2}{9}$, for E_λ $={\cdot}985$, and assuming μ has the same value in both cases $\mu E_\lambda=1{\cdot}77\times10^{-18}$. Thus Ca-atoms radiating $\lambda=4227$ behave like atoms of weight ·9. The level reached should therefore be less than half the level reached by the H, K lines. The actual value is 5000 *km*.

For H-atoms in the state (1), the absorption band for the normal atom lies at $\lambda=1216$, the value of $E\lambda$ is only $\dfrac{5{\cdot}3\times10^{-3}}{998\times10^{-3}}$ or $\dfrac{1}{188}$ times the emission for the Calcium bands H-K. Whatever may be the value of μ for Hydrogen, it must be of the same order of dimension as the corresponding quantity for Calcium ; if we take them to be equal, the compensation is equivalent to a reduction of $\dfrac{40}{180}=\dfrac{2}{9}$ of the gravity, so that Hydrogen atoms in state (2) behave as if its atomic weight were reduced to ·778. It will thus be seen that the level reached by Hydrogen atoms in state (2) will be much less than the radiant atoms of Calcium emitting H. and K.

Similar calculations may be carried out in the case of Helium, Strontium, and Barium, but owing to the great uncertainty in our knowledge of the value of μ, the calculations will scarcely appear convincing. It is easily seen that a slight variation in the value of μ may cause the force of levity vary within very wide limits. The calculations given are therefore to be regarded only as provisional, and we must wait for further theoretical, and practical work before a final shape can be given to the theory.

There has been, in the past, a very keen controversy about the occurrence of Radium in the Chromosphere of the Sun. In 1912 Dyson[1] identified four Chromospheric lines $\lambda=3814{\cdot}7$, $4682{\cdot}4$, $3649{\cdot}8$, $4436{\cdot}5$ as being identical with the four most prominent emission-lines of Radium obtained by Runge and Precht. Of these, the identification in the case $\lambda=3814{\cdot}70$, $4682{\cdot}36$ were most convincing. But the former was generally ascribed to Titanium, and the latter to either Yttrium or Cobalt. The Chromospheric level reached by these lines are respectively 800 km. and 350 km. (Mitchell. Astr. Journal, Vol. 38).

[1] Observato y, Vol 35, pp. 297, 357, 402.

The identification was however disputed by Mitchell and Evershed, who regarded that the coincidence was not quite convincing. In fact, the Ra-atom is so heavy (At W-226) that it appeared unlikely that it could occur at such a height.

The question may be reopened from our point of view. Radium belongs to the group of Alkaline Earths,—(Ca, Ba, Sr),—elements which are marked by the abnormal height of their principal pair-lines in the Chromosphere. The lines (4682 and 3815) are the leading-members of the Principal-pair series of Ra corresponding to the H and K of Calcium. The compensation due to radiation-pressure is therefore likely to be very large, much more so than in the case of Barium (At W-137, the Principal pair are 4934 and 4554), the heaviest of the alkaline earths occurring in the Chromosphere (Height reached 750—1200 kms.). The shorter line 3815 is the more intense, and reaches a much higher level, as is generally the case with the corresponding lines of the Ca, Sr, Ba. The occurrence of Radium in the solar Chromosphere is therefore quite probable.

§ Spots and Prominences.

In recent years, our knowledge of sun-spots and prominences has been greatly increased, thanks to the works of Evershed, Adams, Hale, St. John and others. At the same time, more problems have been brought to light than solved.

Among these problems may be mentioned the problem of the formation of a pseudo-vortex over the spot in high level Chromosphere. According to modern theories, a sun-spot is a cyclonic vortex in the deeper strata of the Reversing layer. The vortex is in the shape of a hyperboloid of one sheet, with its axis vertical, the stream lines being in the form of nearly horizontal helices with gradually increasing radial motion in the upper parts. The temperature of the umbra is believed to be not greater than 3500°C.

The discovery that gases flow radially outwards parallel to the surface of the solar disc from the centre of the spot is due to Evershed. St. John investigated the effect in the case of the different Chromospheric lines, and found that it varies in a regular manner with the level of the line as shown by the flash spectra. In the Reversing layer, the motion is outwards, but as we gradually pass upwards, the radial outflux decreases till it almost vanishes as we enter the Chromosphere. Higher up in the Chromosphere, the effect is reversed, gases now flowing radially inwards with small downward components. The inward motion increases with the height of the line, amounting to about 4 km. in the case of the Calcium H-K lines.

Apparently a pseudo-vortex is formed in high level Chromosphere, with an influx and sucking of the Chromospheric elements into the spot. St. John[1] has adduced rather convincing arguments that the

[1] St. John, the Astro. Journal, Vol. 37.

[2] Evershed, Astro. Journal, Vol. 25, 1909.

[3] Hale, Astro. Journal, Vol. 28, 1908.

[4] St. John, loc. cit., p. 342.

pseudo-vortex cannot be conceived as a sort of compensating device to make up for the general loss of matter from the spots, because the masses, and energy involved in the two cases are entirely of different orders of magnitude.

The theory of "Selective Radiation Pressure" probably affords a satisfactory basis for the explanation of these pseudo-vortices. We have seen that a spot is a region of low temperature, and therefore, of low emissivity. Under circumstances of no disturbance, the gravitational attraction on the high level chromospheric elements is almost neutralized by the aggregate pressure of radiation from the undisturbed disc. If now the emissivity of a part of the disc be decreased much below the normal value, the gravitational acceleration will no longer be balanced by radiation pressure. Equilibrium will be destroyed, and the gases will move towards the spot as a centre of attraction. The effect will be most marked in the case of elements for which neutralisation is almost complete. The hypothesis of "Selective Radiation Pressure," therefore, satisfactorily accounts for the influx, and sucking of the gases lying over a spot.

Prominences afford a rather opposite class of phenomena. As is well-known, these are huge jets of glowing gas, mostly composed of high level Calcium, or Hydrogen, and having their origin in the lower strata of the Chromosphere.

Prominences are generally classified under two broad headings, *viz.*,—the quiescent ones, and the eruptive ones. The eruptive prominences are huge masses of high level Chromospheric elements which spring into existence with the rapidity of an explosion, and move radially outwards with an accelerating velocity, and dissipate into space after a brief career. The quiescent prominences seem to be of the nature of huge cloudlike formations, hovering over the disc, and partaking in its general motion. These prominences show a eleven-year cycle like the spots.

The high velocity, and the explosive violence of the eruptive prominences have been long noticed by the Astrophysicists. Deslandres[1] describes a prominence which he observed in 1898, which was found to shoot up with a velocity of $4{\cdot}34 \times 10^{7}$ cm. per second.

[1] Quoted by Strutt, M.N.R.A.S., Vol. 73.

Evershed[1] describes a prominence observed at Kodaikonal which was found to shoot at $8^h\ 21^m$ with a radial velocity of 79 km. The velocity gradually increased and amounted to 292 km. at $8^h\ 55^m$, *i.e.*, the prominence moved with a mean acceleration of $\frac{1}{6}$ km. ($\cdot 4 \times$ the solar gravity) per sec.[1] It thus appears that in this case, the value of gravity was not only neutralized, but rather reversed, by the sudden development of some repulsive force on the solar disc.

Describing this prominence, Evershed[2] says —

"There is some evidence that eruptive prominences consist in their earlier stages of unusually dense low-lying gas giving strong absorption in H-α and the Calcium H, K. The mass of gas may persist for several days apparently unchanged, and then become unstable, coming under the influence of a force which apparently tears it to shreds, and sends the fragments flying into space with accelerating speed."

Deslandres[3] has also found that all eruptive prominences have 'filaments' (streams of convection lines within the sun) as their bases.

While the phenomenon has been studied under all its details, the explanations from the physical stand point have, as usual, lagged much behind. They have sometimes been explained on the assumption that they are due to the convection of hot masses of vapour from the interior of the sun which, after reaching the atmosphere, are supposed to expand adiabatically and develop the large velocities which are observed. But both Pringsheim and Strutt[3] have pointed out insuperable difficulties in the way of the acceptance of this hypothesis, including the observations that this *great velocity is acquired outside the photosphere*, and the *maximum velocity obtainable from adiabatic expansion is only* 1/60 *of the velocity* with which the eruptive prominences are observed to shoot up. Strutt has suggested that some unknown forces of electrical origin may be the cause of these large velocities, but even granting electrical fields exist in the sun, it is difficult to see how it can act upon the luminous hydrogen particles which are, according to laboratory evidence,

[1] Bulletin of the Kodaikanal Solar Observatory, No. 51.

[2] Evershed, loc. cit., p. 214.

[3] Pringsheim, Physik der Sonne, page 225.
Strutt, M. N. R. A. S., Vol. 73.

uncharged. Besides, if the electrostatic fields really exist, the H-lines should afford evidences of Stark effect. But all observations to this effect have yielded only negative results.

The 'accelerating speed' of the eruptive prominences can be readily explained on the basis of the present theory. The equilibrium temperature of the solar disc is 7500°K, but the temperature equilibrium is not always maintained, but is sometimes violently disturbed by the arrival of hotter gases in the form of convection streams from the interior (Deslandres calls them filaments).

Suppose somehow the temperature of a part is increased owing to the arrival to hotter gases from the interior beyond the normal value (7500°K), say to 9000°K. Then the emission at the absorption lines of Calcium, and Hydrogen will be much more increased, and the repulsive force will greatly preponderate over the normal force of gravity. For example, if the temperature increases to 9000° K, we have, applying Wien's law of radiation (E_λ=emissivity for 7500° K, E'_λ=Emissivity for 9000°K, λ=3920, the wave of maximum emission for 7500°K),

$$\log_e\left[\frac{E'_\lambda}{E_\lambda}\right]=\frac{c_1}{\lambda}\left(\frac{1}{\theta}-\frac{1}{\theta'}\right)=\frac{c_1}{\lambda\theta}\,\frac{\theta'-\theta}{\theta'}=\frac{\beta}{6}\,\beta=\frac{c_1}{\lambda\theta}=5 \text{ (approximately)}$$

$$\text{or } E'_\lambda=e^{\frac{5}{6}}\,E_\lambda=2E_\lambda \text{ (approximataly).}$$

Now we assumed that the reduced gravitative force on an atom of Calcium=$(Mg-\mu E_\lambda.)$

It now becomes=$(Mg-2\mu E_\lambda.)$

Hence if $Mg-\mu E\lambda=0$, $Mg-2\mu E\lambda=-Mg$, *i.e.*, the Calcium atoms emitting H and K which are just over the heated region are repelled outwards with an acceleration of $(-g)$.[1]

We have seen that generally the eruptive prominence, observed by Evershed, moved outwards with a negative acceleration amounting to 1/9g. This can be accounted for by assuming a temperature of 8500°K just below the prominence, over a limited region of the solar disc.

There is nothing to preclude the supposition that occasionally the temparature equilibrium of the solar disc is disturbed. It has

[1] The calculation applies only to the gases lying at a short distance over the temporarily heated part. For more distant masses, we must take into account the solid angle subtended by the heated spot at the point considered.

been shown beyond doubt that spots are not at higher temperatures than 3700° C. or 4000° K. The faculae and filaments are generally brighter than the rest of the surface, and therefore regions of high temperatures. We have seen that according to Deslandres, these act as the bases of eruptive prominences. According to our theory, these heated parts being regions of high emissivity, should act as centres of repulsive force. The explanation is therefore quite in accordance with observed facts. Besides, it explains most satisfactorily the fact noticed by Strutt and Evershed that the velocity is acquired outside the photosphere, and is of the nature of accelerated speed.

BOTANY

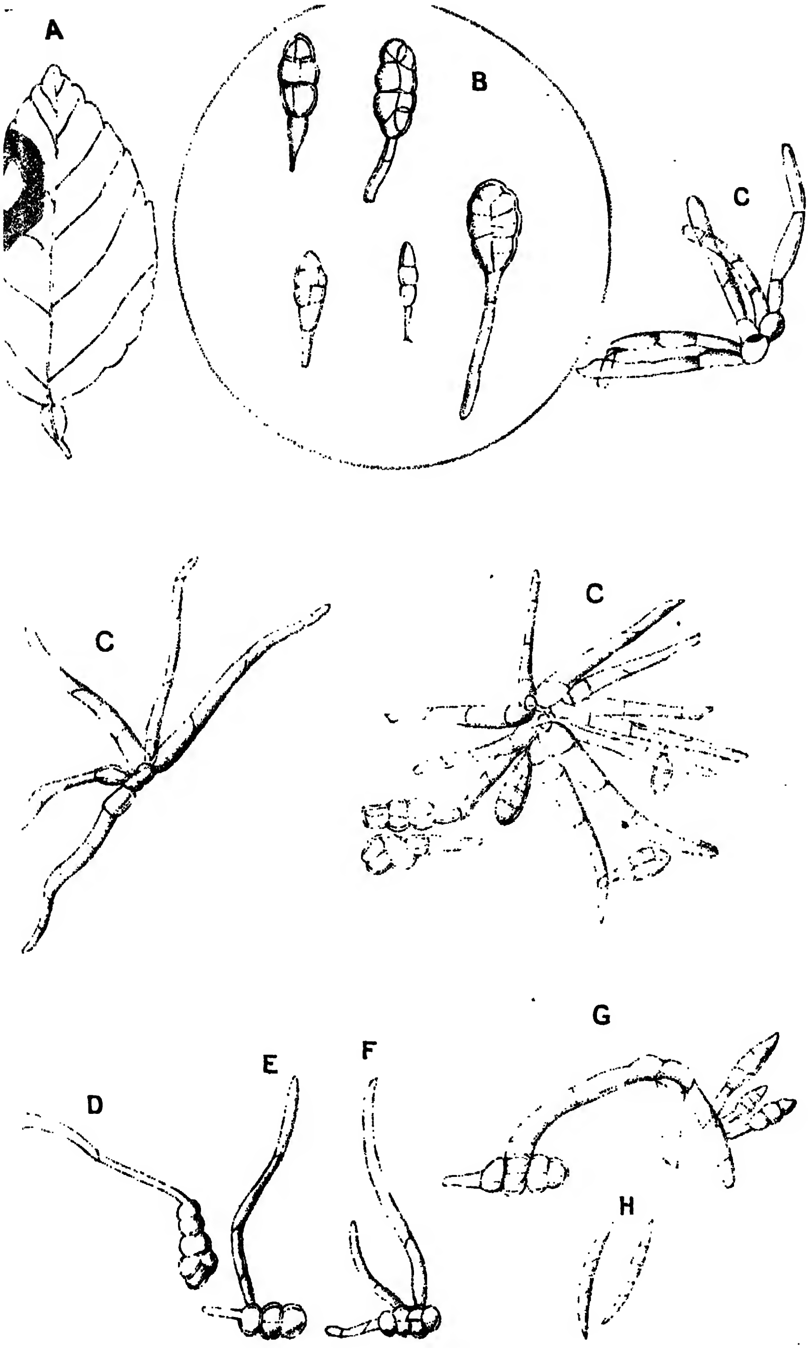

Plate illustrating *Macrosporium* on leaves of *Citrus medica.*

Commentationes Mycologicæ.

2

Macrosporium (FRIES) GROWING ON *Citrus Medica* (VAR. *acida*) AND OTHER SPECIES OF *Citrus*.

BY

S. N. BAL.

The Fungus described below was first noticed on the leaves of *Citrus Medica* (var. *acida*) in the middle of May of the present year and is still found. Later on the same fungus was observed growing on the leaves of other species of *Citrus*.

External appearance :—At first a small portion of the margin or the middle of the leaf is attacked, but the infested area soon increases in size. The leaf becomes yellow, and the infested parts dry and slightly curl up, whilst small black spots make their appearance and the centre of the attacked area becomes white and finally the leaf drops off.

On examination of the infested portion of the leaves under low power it is seen that numerous hyphae have spread out from the base of old conidiophores and most of these hyphae termimate in conidia of different sizes and of a brownish colour. The hyphae are 0·12—0·18 mm. long and they are either straight or slightly curved.

On examination of transverse sections passing through the infested area of a leaf, it appears that the fungus growth is both superficial and intracellular. The superficial hyphae are dark-coloured in the vicinity of the conidia and those situated within the leaf tissue are colourless and thinner. The small twigs bearing the infested leaves were also found to be infested with the mycelia of the same fungus. It is, therefore, evident that the mycelium penetrating deeper into the tissues of the twigs may easily plug the water-conducting parts causing the collapse of the leaves and the shrivelling up of the flower-buds on those twigs. The actual damage done to the plant, under observation, was (1) the decrease in the number of fruits; (2) the reduction of the size of the fruits.

The Conidia are brown, multicellular, clavate, variously muriformly septate, the thicker portion being oblong or ellipsoidal with the apex rounded or subacute. The size of the Conidia is as follows:—

Young Conidia	...	45-60μ long and 8-10μ broad.
Mature Conidia	...	80-100μ long and 12-18μ broad.

Germination of the Conidia was effected in hanging drop culture, using tap-water. The Conidia commenced germinating after 48 hours and in 5 days' time a mycelium, bearing several Conidia, was produced. (See the plate attached to this paper.)

It is to be noted that scattered spores of *Fusarium* were also observed on examining the fungus above described, but the spores were clearly foreign to the mycelia described above.

The disease caused by this fungus has been described by other observers as Potato 'leaf curl' and 'black rot' of Tomato, but it does not appear to have been described before on leaves of species of *Citrus*. Some observers have found the same fungus on Tomato fruits causing the disease commonly known as 'black rot' and named it *Macrosporium tomato*.

The fungus, above described, is evidently identical with or closely allied to *Macrosporium solani*, Cooke. 140 species of this fungus have been described from Europe and America. *Macrosporium solani* has been described from North America on *Solanum tuberosum* and in India it has been found both on Potato and Tomato plants.

The remedy to the disease seems to be to collect all the infested portions as soon as noticed and to destroy them.

Explanation of the Plate.

A. A leaf of *Citrus medica* (var. *acida*) infested by the fungus—natural size.

B. Conidia of different sizes and forms, × 430.

C. Hyphae bearing conidia × 430.

D. A single hypha bearing a conidia × 430.

E. Germinating conidia as seen after 2 days' culture × 430.

F. Germinating conidia as seen after 3 days' culture × 430.

G. Germinating conidia as seen after 5 days' culture × 430.

H. Spores of *Fusarium* foreign to the fungus described × 430.

COMMENTATIONES MYCOLOGICAE.

3

Exoascus (Fückel) on *Nephelium Litchi*.

BY

S. N. BAL

The fungus, described below, was first noticed on the leaves of *Nephelium Litchi* during the early part of April of the present year and is still found, though not so abundantly as before. The trees were also watched during the winter of 1918-19, when no outward attack was observed on the leaves.

External appearance:—At first it develops on the under surface of the leaf and soon attacks a greater portion or the whole of the leaf. The diseased leaves are puckered, crumpled or twisted. The more vigorously attacked leaves are so deformed that the margins of the leaves are gradually turned towards the midrib forming a spindle-shaped body and in other cases the leaves are curled up. The infested leaves are very much thickened and become quite fleshy. At first the infested portion becomes pale green and then turns to brownish yellow, finally turning to deep brown.

The twigs bearing the diseased leaves were swollen at parts which were subsequently found to be attacked by the fungus.

On examination of transverse sections passing through the infested portions of leaves it appears that the fungus is chiefly composed of a single pallisade-like layer of asci standing close beside one another. The asci break through the cuticle and cover the outer surface of the epidermis of infested portions of the leaf. This layer of asci is developed from the mycelium lying in the inner parenchyma of the part attacked, thrusting its branches in between the outer walls of the epidermal cells and the cuticle. At this stage the mycelium branches and spreads out on the surface forming a single layer and dividing themselves into isodiametric cells. Each of these cells thus formed swells into a vesicle and elongates towards the surface forming a club-shaped cell. This cell again divides transversely forming a short *stalk-cell* at the base and a bigger cell at the upper portion forming the club-shaped *ascus*.

The mature ascus generally contains 8 ascospores, but these spores are also found to germinate inside the ascus forming the so-called budding spores, filling up the whole ascus. The ascus is 40-50μ long and 8-10μ broad.

On examination of transverse and longitudinal sections of twigs and branches bearing the infested leaves it appears that they are also attacked by the fungus, and the mycelium penetrates through the intercellular spaces of the tissues of such portions of twigs and branches. As the attack was not observed during the winter season, but appeared during the spring-time, it is quite possible that the mycelium hibernates in the stem during the winter, and sends branches every year, during spring-time, to the new twigs and branches and thence to the leaves to form asci and thus disfiguring the leaves.

Germination of mycelium and spores was effected on nutrient agar plates and in tap-water respectively. By germinating the mycelium on nutrient agar plates it was possible to produce asci. The spores were found to germinate in 8-12 hours in tap-water, sprouting repeatedly and perfectly. The sizes of the spores differ in each order of sprouts. Those of the first order are practically of the same size as the mother spores and those of the higher orders are seen to be much smaller. They are all oval or spherical.

The sizes of the spores are :—

Mother-spores and spores of the 1st order are 6-8μ.

Spores of the higher order are 4-6μ.

The fungus is evidently identical with or closely related to *Exoascus deformans* (Berkeley) Fuckel, as described on page 160 of *Abteilung* I of the 1st part of Engler and Prantl's *Pflanzenfamilien*.

The synonymy of *Exoascus deformans* (Berk.) Fuckel has been given by numerous writers. Dr. R. Sadebeck, in his "Die parasitischen Exoasceen," Hamburg. 1893, p. 53 gives it as follows :-

Ascomyces deformans Berk. Introd. to Crypto. Bot. 1857, p. 284.

Ascosporium deformans Berk. Outlines 1860, p. 449.

Taphrina deformans Tul. Ann. Sci. Nat. 1866, V. Ser. t. 5., p. 224.

Exoascus deformans Fuckel. (*a*) *Persicae* Fuck. Symbolae Micolog., 1869, p. 252

It would not be out of place to mention here that the decease commonly known as "Peach leaf curl," which caused serious damage to the Peach trees grown in the United States of America, was due to the very same fungus *Exoascus deformans*. The investigations carried out in this connection by Pierce of the U. S. department of Agriculture in 1900 showed that a cold spell in the spring and wet weather favour

serious development of the fungus causing the desease and result in a much greater susceptibility of the tissues of leaves to the attack of the parasite. Prof. A. Marshal Ward in his Croonian Lecture on "The relations between host and parasite in certain deseases of plants," (Proc. Roy. Soc., Vol. XLVII, No. 290), says: "when the combined effects of the physical environment are unfavourable to the host but not so or even favourable to the parasite, we find the desease assuming a more or less pronounced epidemic character." The writer also thinks, as the result of his observations during the last year of the development of the fungi on the leaves and twigs of *Nephelium Litchi*, that the fungus is favoured in both its entrance and spread within its host by the above enumerated physical conditions *viz.*, cold and wet conditions of the atmosphere.

It is quite evident, from the descriptions of the fungus given above, that the fungus is liable to do much harm to the lichi crop, and in fact it has been observed that the few trees under observation bore a much lesser number of fruits than usual.

The remedy suggested by Massee seems to be the best. He suggests that the only certain method of eradicating this disease is by removing all infested shoots. Many advocate spraying with Bordeaux Mixture. It will undoubtedly prevent infection by spores, but the perennial mycelium present in the shoots will produce a crop of diseased leaves every year inspite of spraying, and, if the cause of infection is removed by cutting away infected shoots, no spore would be forthcoming to infect healthy shoots.

Explanation of the plate :—

A. Leaves of *Nephelium Litchi* infested by the fungus—Natural size.

B. A leaf of *Nephelium Litchi* vigorously attacked by the fungus almost totally curling the leaf—Natural size.

C. Transverse section of infested portion of a stem × 16, (*a*) portion of C showing the infested portions.

1. A layer of asci × 420.
2. An ascus obtained by culture in nutrient Agar plate × 420.
3. A portion of the transverse section of infested portion of a twig showing the penetration of mycelium through the intracellular spaces, × 440.
4. Germinating spores × 440.

Commentationes Mycologicae.

4

Alternaria Nees, on leaves of

(*a*) *Nicotiana plumbaginifolia*,

and

(*b*) *Datura Stramonium*.

BY

S. N. Bal and H. P. Choudhury.

(*a*) *Alternaria* on *Nicotiana plumbaginifolia*.

(By S. N. Bal and H. P. Choudhury).

The fungus described below was first noticed on the leaves of *Nicotiana plumbaginifolia*, which grows wild in most parts of Bengal. At first the plants grew vigorously, attaining a height of $2\frac{1}{2}'$ to $3'$, and produced very large leaves, some of which were more than 5" in diameter. The plants were watched for some time and within a fortnight after the attack was first observed most of the plants died During the early part of June of the present year there were a few showers of rain and a fresh crop of *Nicotiana plumbaginifolia* sprang up from the seeds of plants of the previous crop. This new crop of plants was soon attacked by the same fungus and most of them died before they attained the height of even 3", whilst none of them attained maturity.

External appearance: At first greenish yellow spots appear at various portions of the leaf. These spots soon extend in area, assuming a semi-transparent appearance and the infested portions of the leaf fall off, leaving a hole at the centre. Within 15 days after the first attack the major portion of the leaf is infested and it soon falls off.

On examination of portions of leaves infested by the fungus the spots are found to be full of spores. Short conidiophores bearing either single or a chain of conidia are also found. The mycelium is seen to consist of light brown or greenish hyphae running in between the cells at the spots. Conidiophores are seen in clusters coming out through the stomata or killed epidermal cells.

The spores are usually attenuated into a long beak at the tip and slightly narrowed at the base and they are transversely divided by 3—8 cross-walls, and some of the broader compartments are further longitudinally septed: the lower part of these spores is brown and the neck is almost colourless. They are surrounded by a thin cell-wall and filled with granular substance.

Germination of the spores was readily effected in tap water. Single spores were germinated in hanging-drop culture, keeping them overnight in sterilised moist chamber. Secondary conidia were abundantly produced from the primary spores, the conidiophores frequently producing conidia borne in a chain of 4 to 6 spores. The spores are 60—80μ long × 10—15μ broad.

The fungus described above is identical with or closely related to *Alternaria violae* Dorsett as described by Dorsett in Bull. No. 23 (1900) of the United States Dept. of Agriculture; Pathology and Physiology Division. The disease is said to be one of the most widespread and destructive maladies known to attack the violet in the United States.

(*b*) *Alternaria* on *Datura Stramonium*.

(By S. N. Bal).

The fungus was noticed on the older leaves of *Datura Stramonium* during the early part of May of the present year and is still found in July.

External appearance: At first it appears on the leaves as small isolated, pale brown spots, there being no particular portion of leaf which can be called "the attacked". The infested portions of leaf soon increase in size, becoming irregularly circular showing concentric, narrow dark lines, giving the appearance of a Potato starch grain as seen under the microscope. At the advanced stage of the disease, the spots become dark-coloured, dry and brittle. The portions of leaves between the spots turn slightly yellow and then fade away. In one plant, which was severely attacked, the stalks turned yellow and dried up. In all cases only the older leaves were attacked, while the upper younger leaves remained free from the disease.

On examination of transverse sections passing through portions of leaves infested by the fungus it appears that the mycelium consists of light brown hyphae. The conidiophores arise from the stomata or from the dead epidermal cells. They are mostly found in the centre of the spots. The conidiophores are 60-80μ long × 8-10μ broad. They are transversely septed, sometimes curved and generally enlarged at the tip forming a cup-shaped head on which single conidia are borne.

The conidia vary in shape. They are generally flask-shaped, the apex or the neck being attenuated and the base being slightly narrowed and brown in colour, whilst the apex is almost colourless. The spores were always found single and not in chains, but chains of two and rarely of three spores were obtained in culture. The spores are 140-300μ long × 15-17μ broad. They are transversely septed, and some of the cells are again longitudinally divided.

Germination of the spores was easily effected by hanging-drop culture using tap water. They germinated readily in 24 hours, but the peculiarity observed was that they refused to produce long chains of spores and only formed two and rarely three spores in a chain, thus behaving quite differently in this respect from the species of *Alternaria* found on the leaves of *Nicotiana plumbaginifolia.*

Inoculation experiments were carried out on the leaves of *Solanum Melongena* and was quite successful, the germ-tubes entered the leaf through the stomata or directly into the epidermal cells. The leaves were kept quite moist. Spots appeared on the leaves in 4 days. On subsequent examination they showed the same external and microscopic characters as described for the main fungus.

The fungus is evidently identical with *Alternaria Solani* (E. and M.) Jones and Grout. Butler describes this fungus in his book "Fungi and disease in plants" as "Early blight" of potato. It is said to have done considerable damage to the potato crop in America, where losses of 50 per cent. have been recorded. In India the damage is much less.

Spraying with Bordeaux Mixture gave very excellent results in the United States. Butler says "The disease is not dependent on atmospheric humidity to anything like the extent blight is. It spreads rapidly under relatively dry conditions and is only checked by severe drought. It is said to be much in evidence in Australia on light sandy soils."

"There is great danger in neglecting rotation where this disease is prevalent, owing to the pronounced vitality of the mycelium and spores. Potatoes are often grown year after year in the same land in Bengal and even two crops a year in the Khasi and Nilgiri hills It has been observed in similar cases that the disease gradually increases. Clean removal and burning of infested tops at harvest is strongly recommended for the same reason. If the mycelium hibernates in the tubers, clear seed from a disease-free crops should be used. Some varieties are said to be less damaged than others and their use might be extended."

"As with all other parasites that prefer weakened plants, high cultivation, heavy manuring, and anything that stimulates the crop, lessens its liability to attack."

17 species of *Alternaria* have been described, of which 5 have been reported from Middle Europe. *Alternaria Solani* has also been found to attack the leaves of *Solanum tuberosum* *Solanum nigrum*, *Solanum Commersoni* and *Hyoscyamus niger*.

Explanation of the plate.

A. Seedling of *Nicotiana plumbaginifolia* attacked by the fungus —Natural size.

B. A leaf of *Nicotiana plumbaginifolia* attacked by the fungus leaving holes at the centre of the spots—Natural size.

C. A transverse section of a portion of a leaf infested by the fungus, showing conidiophores bearing conidia—Highly magnified.

1. Germinating spore of one day's culture Highly magnified
2. ,, ,, ,, two days' ,, ,, ,,
3. ,, ,, ,, three ,, ,, ,, ,,
4. ,, ,, ,, two ,, ,, ,, ,,
5. ,, ,, ,, ,, ,, ,, ,, ,,
showing numerous cells germinating.
6. Germinating spore of 5 days' culture × 440.

X. A portion of a leaf of *Datura stramonium* infested by the fungus —Natural size.

7. Spores as seen at the centre of the spots × 420.
8. A germinating spore of three days' culture × 420.

On the Systematic Position of Lindenbergia, *Lehmann*

BY

Dr. P. Brühl.

When, in 1907, botanizing in the Darjiling District, I noticed that the aestivation of the corolla of *Lindenbergia grandiflora*, Benth., is not that generally held to be characteristic of the Antirrhinoideae. Similar observations with respect to other species of Lindenbergia have evidently been made by Theodore Cooke and J. F. Duthie. Cooke, in his excellent and exceedingly useful Flora of Bombay, places Lindenbergia between Sopubia and Centranthera, two closely related genera, members of the Rhinanthoideae-Gerardieae, whilst Duthie, in the Flora of the Upper Gangetic Plain, assigns it a place after Sopubia and states explicitly that the upper lip of the corolla in Lindenbergia is innermost in bud. It was of interest to investigate whether any of the other genera of the Gratioleae—Stemodiinae, with whom Lindenbergia is associated in the Genera Plantarum of Endlicher and of Bentham and Hooker, the Flora of British India and other works, similarly deviated from the aestivation characteristic of the Antirrhinoideae, and for this purpose the writer has examined the buds of various species of Limnophila, Morgania, Stemodia, and Adenosma, with the result that in these genera the posterior lip of the corolla is invariably outermost in bud. Cooke has evidently carried out a similar investigation, as he explicitly states that the upper lip is outermost in bud in Moniera, Mimulus, Stemodia, Limnophila, and Dopatrium. I have further examined buds of *Lindenbergia Hookeri*, *L. philippinensis*, *L. macrostachya*, *L. abyssinica*, *L. Griffithii*, *L. urticifolia*, *L. polyantha* and *L. sinaica*, in all of which the upper lip is innermost in aestivation. This therefore is a constant character of the genus Lindenbergia and suggests an inquiry into the real relationship of this genus.

The salient characters which have to be considered in such inquiry are the aestivation of the corolla, its morphological characters, the nature of the stamens and its fruit. The older botanists—Retz,

Vahl, Roxburgh—do not pay any attention to aestivation, and Endlicher in his Genera Plantarum confines himself to the statement that the aestivation of the corolla of Scrophularinae is imbricate. It was Bentham who in his monograph of the Scrophulariaceae in De Candolle's Prodromus (1847) laid stress on the importance of the relative position in bud of the parts of the upper and lower lip or the corresponding parts of the corolla-limb when divided subequally into lobes. He characterizes the two series of Antirrhinidae and Rhinanthideae as follows:—Anthirrhinidae—aestivatio corollae imbricato—bilobata, *labio postico exteriore.* Inflorescentia vel undique centripeta vel composita, partialis in cyma centrifuga. Rhinantideae—*aestivatio corollae imbricata, labio postico nunquam exteriore.* Rigorously applying his principle of discriminating the two series Bentham places his subtribus Escobedieae with the Antirrhinidae. In Bentham and Hooker's Genera Plantarum the distinction between the Antirrhinidae and Rhinanthideae is less sharply cut, as is evident from the following definitions of the two series:—

Antirrhinidae: Folia saltem inferiora saepissime opposita (in perpaucis generibus pleraque radicalia caulinis alternis). Inflorescentia undique centripeta vel composita partiali centrifuga. *Corollae labium posticum vel lobi postici exteriores, lobo antico intimo* (in subtribu Limosellearum aestivatio incerta). Stamen quintum ad staminodium reductum vel omnino deficiens, rarissime perfectum.

Rhinantideae: Folia varia. Inflorescentia saepius centripeta vel composita, *corollae lobi varie imbricati, antico vel lateralibus saepius exterioribus.* Stamen quintum omnino deficiens vel rarissime perfectum.

It appears, however, that the authors lay the main stress on the relative position of the corolla-lobes in aestivation. Bentham is followed by Le Maout and Decaisne in their Traité Général de Botanique, in which the Antirrhineae and Rhinantideae are respectively defined by "corolle à préfloraison imbriquée, bilabiée, la lèvre postérieure ou supérieure placée en dehors de l'inférieure" and "corolle à préfloraison imbriquée, les deux lobes latéraux, ou l'un d'eux, placés en dehors de tous les autres, les postérieurs jamais." Wettstein in Engler and Prantl's Pflanzenfamilien characterizes the Pseudosolaneae and Antirrhinoideae jointly by stating that the two adaxial corolla-lobes or the upper lip overlap in bud

the lateral lobes, whilst in the Rhinanthoideae the posterior corolla-lobes or the upper lip are overlapped by one or both the lateral lobes. All these authors place Lindenbergia in the series of Antirrhinoideae. The same position is assigned to Lindenbergia in the Flora of British India, where it figures at the head of Stemodieae, a subtribe of Tribe VI Gratioleae, and there associated with Adenosma, Stemodia and Limnophila. Boissier in the Flora Orientalis places Lindenbergia between Dodartia and Herpestis among the Gratioleae, but does not refer to the aestivation of the corolla in his definition of his tribus I to VI, although he states that in Gerardieae (tribe VII) the posterior corolla-lobes are most frequently innermost and in Euphrasieae (tribe VIII) the upper lip is innermost. If we ignore the aestivation of Lindenbergia, there is not the slightest doubt that its proper place is in the Gratioleae, and if it were not for its separate and distinctly stipitate anther-cells, it should find its place close to Mazus and Mimulus in Subtribe Mimuleae. But if we leave it among the Antirrhoideae, we are faced with the fact that in that case a character shared undoubtedly by all the other genera of that series, namely the upper lip being outermost in bud, is consistently the opposite in Lindenbergia. It is perhaps not a matter without significance that the corolla, at least of all the Indian Lindenbergias, is yellow, whilst that of the other genera belonging to the Stemodiinae is blue; Boissier, however, states that the corolla of Lindenbergia sinaica, a species closely related to Lindenbergia polyantha, is "violacea," a matter which might be more closely investigated by botanists who have the opportunity of botanizing in the vicinity of Aden or on the Sinai Peninsula.

If, then, we decide on removing Lindenbergia to the Rhinantoideae, the question is in which of the three tribes—Digitaleae, Gerardieae and Euphrasieae—it is to be assigned a place. Every botanist will agree that Lindenbergia has absolutely no close relationship with any of the genera which go to form the tribe of Digitaleae. If we should decide to place it in the Girardieae, there is no doubt that the only subtribe concerned is the subtribe Eugerardieae. And that is evidently the reason why Theodore Cooke placed it between Centranthera and Sopubia. To arrive at a valid conclusion in this respect the writer has made a detailed study of the specimens representing the genera Centranthera, Sopubia, Pseudosopubia, Buttonia, and Graderia in the Sibpur herbarium. The parts of chief interest

are the stamens and pistils, particularly the former. The species of Sopubia in which the anther-cells are practically alike is *Sopubia stricta*. In this species the thecae are equal in size and shape being obovate, narrowed at the base and apiculate at the tips, more or less divergent, about 1·5 mm. long and opening by two flaps and after dehiscence somewhat resemble miniature pods; they are inserted on the slightly thickened apex of the filaments, are more or less pendulous and form various angles with each other. Unfortunately the anthers of all the specimens examined had shed their pollen; but, to judge from appearances, both thecae of each anther had been polliniferous. The thecae are certainly not stipitate. As far as the anthers are concerned the relation of *Sopubia stricta* to *Sopubia delphinifolia* is very much like that of *Graderia scabra* to *Buttonia natalensis*. The first step in the direction of modification of the anthers is that shown by *Sopubia trifida*, Ham. Here the second theca of each anther is considerably reduced and probably sterile, although examination of fresh specimens may prove that sterility is not always complete. The larger thecae are about 1·8 mm. the smaller 0·5 to 0·8 mm. long; the latter also open by a slit and look like smaller copies of the fertile ones. The specimens examined were from Cherrapunji and the Naga Hills. Undoubtedly sterile are the smaller thecae of *Sopubia dregeana*, Benth., *Sopubia cana*, Harvey, and *Sopubia lanata*, Engl., all three African species. Here the sterile thecae are narrow-spatulate or flattish clavate; they are distinctly shorter than the fertile ones and in one species are only half the length of the latter. Next in order come *Sopubia delphinifolia*, Don, and, what may be only a variety of the latter, the Indo-Chinese *Sopubia fastigiata*, Boneti. Here the polliniferous thecae are 3—4, the sterile 3 mm. or less long; the sterile thecae are adnate at their base on one side for about 1 mm. to the filament; their shape is subulate with a median groove; the anthers of the anterior pair of stamens seem always to be free from each other; the fertile thecae of the pair of posterior stamens are joined together on the inner side by a dense fringe of short curly hairs, whilst the corresponding sterile thecae are probably in most or all cases connate for part of their length, leaving the pointed tips free. The final stage is reached by *Sopubia Hildebrandii*, Vatke, an East African species, now removed to a new genus Pseudosopubia. Here for once the sterile thecae are distinctly stipitate and reduced to a small knob, whilst the fertile thecae are raised considerably above

the sterile ones on a filiform stipe. The sterile thecae of one pair of stamens are entirely reduced. I may add here that fig. 41 E of Part III, 3 B of Engler and Prantl's Pflanzenfamilien shows one of the stamens not of *Sopubia delphinifolia*, but evidently of *Sopubia trifida*.

A considerable amount of variation is also discernible in the stigma of different species of Sopubia.* That of *S. delphinifolia*, *S. fastigiata* and *S. Hildebrandii* is subcapitate, the style at its very apex widening into the short thickly two-lipped terminal stigmatic surface, somewhat like that of *Lindenbergia polyantha*, whilst it is sublinguiform or compressoclavate in *S. trifida*, *S. stricta*, *S. cana*, and *S. dregeana*; a further development is noticed in *S. lanata*, in which the stigma is elliptic-lanceolate; the two nerves which run up the style submarginally in the forms with linguiform stigmas, become in *S. lanata* divergent at the base of the stigma-branches, take their course intra-marginally and converge towards the apex. The capsule of Sopubia cannot be utilised to decide any question of relationship between Lindenbergia and Sopubia.

The conclusion at which we arrive is that the relationship between Lindenbergia and Sopubia is not at all close; and as Centranthera and other Gerardieae are still more distant, Lindenbergia cannot be placed among the tribus Gerardieae.

The only alternative left is to assign it a place in the Rhinantheae, between which and the Gratioliéae it appears to form a connecting link. It is true that many Rhinantheae are root-parasites, whilst any idea of any of the species of Lindenbergia being a root-parasite is out of question. It is also true that as regards the stamens, Lindenbergia appears to stand closer to the genera which constitute the subtribe of Stemodiinae; but the anthers of certain Rhinantheae exhibit various peculiarities which suggest that no violence is committed by associating Lindenbergia with them. On the other hand the structure of the corolla, which might suggest a relationship to some Mimuleae, such as Mimulus, the personate corolla and the plaited lower lips brings it close to several typical genera of the Rhinantheae. As

* *The stigma of Sopubia is correctly described in the Flora of Tropical Africa as thickened or flattened at the apex.*

Lindenbergia has 4 perfect stamens with the thecae fertile and similar, a loculicidal capsule, placentas bearing a great number of ovules, and ebracteolate flowers, it may find its place somewhere in the vicinity of Euphrasia, or it may be placed at the head of Rhinantheae, as in some respects it occupies a rather isolated position.

BOTANICAL DEPARTMENT,
UNIVERSITY COLLEGE OF SCIENCE,
The 1st December, 1919.

NOTE ON LINDENBERGIA URTICIFOLIA, *Lehm.* AND LINDENBRGIA POLYANTHA, *Royle*

BY

DR. P. BRÜHL

Theodore Cooke, on page 307 of Part II of the Flora of Bombay says under *Lindenbergia polyantha*: "Very close to *L. urticaefolia*, of which it may possibly be a starved form." The latter statement is not quite intelligible, as in the detailed description of *L. polyantha* the stem is said to be 12 to 20 inches long, whilst the stem of *L. urticifolia* is stated to vary in length from 4 to 20 inches. Father E. Blatter and Prof. Hallberg, in their valuable paper entitled "New Indian Scrophulariaceae and some notes on the same order"* express the opinion that *Lindenbergia polyantha* does not even deserve varietal rank. They enumerate and describe ten different forms, of which form (8) is, according to these authors, the form described by Cooke as *L. polyantha*.

In this connection it is of some interest to look into the literature referring to either one or the other of the two supposed species. Josephus Gaertner, in his admirable work "De Fructibas, etc.," published in 1788, describes and illustrates the fruits and seeds (page 243 and tab. 52, fig. 5) of a plant named by Banks *Stemodia ruderalis*. A. I. Retzius in "Fasciculus observationum botanicarum quintus," published in 1789, describes under the name of *Stemodia ruderalis* a plant from a specimen sent from India by Koenig. This is accepted in the Index Kewensis as a good species of Stemodia.

Retzius' description runs as follows: "Stemodia ruderalis, foliis ovatis serratis petiolatis. Caulis suffruticosus videtur, teres, pubescens, ramosus. Folia apposita, ovata, serrata: serraturis magnis, basi integra. Petioli tenues, longitudine fere foliorum. Flores axillares, solitarii, oppositi, pedicellati. Calyx campanulatus, quinquedentatus, pubescens. Corolla aurea. Stylus corolla brevius." Two points in this description are against the plant described being a species of Stemodia, namely the calyx being 5-dentate and the corolla being yellow. The Indian Stemodias have a 5-partite calyx and a blue corolla. Retzius' description applies quite well to certain forms of what Father Blatter and Prof. Hallberg call *Lindenbergia urticaefolia*.

* *Journal of the Bombay Natural History Society, Jan. 15, 1918.*

Vahl, in *Symbolae* II, p. 69, referring to Retz, Obs. fasc. 5, p. 25, and Gaertner's De Fructibus, defines and describes *Stemodia ruderalis* in the following terms: "Stemodia foliis oblongis petiolatis, floribus axillaribus oppositis. Caulis herbaceus, erectus, spithameus, obscure tetragonus, pubescens, inferne ramosus; rami simplicissimi, patentissimi, caule breviores. Folia opposita, petiolata, pollicaria v. minora, obtuse serrata, dentata, venosa, nuda, obtusa, basi acuta, integerrima. Petiolus filiformis, foliis brevior. Pedunculi axillares, solitarii, uniflori, lineares, pubescentes. Calyx pubescens." This is referred in the Index Kewensis to *Lindenbergia urticaefolia*. Vahl's plant is undoubtedly a Lindenbergia; but, to judge from Gaertner's figure of the corolla of the species referred by him as *Stemodia r deralis* of Banks, the latter may not be identical with any Lindenbergia, in which case it certainly belongs to the Stemodiinae. The lips of the corolla are depicted as widely gaping; that, however, may be due to the flowers having been pressed, which process often produces such an effect, and in an old drawing by one of the Sibpur artists the corolla of a Lindenbergia is figured exactly as it is in Gaertner's work.

Roxburgh (Flora Indica, III, 95) describes a Lindenbergia under the name of *Stemodia ruderalis*, quoting Willdenow III, 345, etc. Willdenow's description is practically a copy of Vahl's. I shall again refer to Roxburgh's description later on.

Don, in his Prodromus nepalensis, published in 1825, describes under the name of *Stemodia muraria*, Roxb., a Nipal species of Lindenbergia as follows: "Radix caespitosa, fibrosa. Caules numerosi, erecti, palmares, villosi, ramosi. Folia fere Calaminthae, opposita. Flores axillares, subsolitarii, breviter pedicellati, pallide rosei. Corolla villosa, calyce duplo longior." It will be noticed that the description is deficient and the statement that the flowers are rose-coloured is rather startling, as *Lindenbergia urticifolia*, like other Indian Lindenbergias have the corolla-lobes deep-yellow; as, however, parts of the corolla of both *L. urticifolia* and *L. polyantha* are purple or purplish-brown, the expression used by Don may find its explanation. In any case it may be taken for granted that the plant referred to by Don is *Lindenbergia urticifolia*.

In 1828 Link and Otto published in their Icones Plantarum Rariorum Horti Berolinensis a description of the species to which they gave the name of *Lindenbergia urticaefolia*, and they figured it on plate 48. Unfortunately the part containing the description and

illustration is wanting in the library of the Sibpur Botanic gardens. I shall, therefore, in what follows assume that the plant described and figured is identical with that going under that name in the Flora of British India.

Bentham, in his Scrophularineae indicae, introduces for the first time into botanical literature Royle's *Lindenbergia polyantha*. His description is as follows: "Erecta vel ascendens, annua, villosa, foliis ovatis, floralibus inferioribus conformibus, superioribus calyce brevioribus, racemis multifloris, floribus oppositis secundis, corolla calyce duplo longiori." He gives the banks of the Jamna near Delhi as the locality. Of *Lindenbergia urticaefolia* Bentham says: "Erecta vel ascendens, annua, villosa, foliis ovatis, floralibus conformibus, floribus solitariis axillaribus, corolla vix triplo longioribus"; he distinguishes a variety *β major* "caulibus elongatis ramosissimis" and states that *L. urticifolia* occurs nearly all over India, but particularly in the hilly parts at Mussooree, in Nipal, at Hardwar in Oudh, Silhet and Burma, whilst *var. β* is stated to have been gathered by Griffith in Martaban and according to Wallish has been collected on the banks of the Irawaddi. Bentham also remarks that *L. polyantha* of Royle is related to *L. urticaefolia*, but distinguishable from it by its inflorescence.

In his monograph of the Scrophulariaceae, in De Candolle's Prodromus (published in 1847), Bentham states that in *L. polyantha* the racemes are many-flowered, the flowers are opposite and secund and the ovary is glabrous, and he adds: Affinis *L. urticifoliae*, sed imprimis inflorescentia distincta. Corollae glaberrimae labium superius lato-ovatum medio concavum breviter bilobum; under *L. urticaefolia* he says that the leaves are long-petioled, the flowers solitary and axillary, and the ovary glabrous; and he adds: "Herba ramosissima, semipedalis ad pedalis, ramis fere a basi floriferis. Corolla 4-5 lin. longa, glaberrima."

According to the Flora of British India the characters distinguishing *L. polyantha* and *L. urticaefoliae* appear to be in the former having leaves $\frac{1}{2}$, rarely $\frac{3}{4}$ inches long, with the petioles usually very short, the flowers in axillary and terminal leafy spikes, flowers and bracts being crowded, whilst in *L. urticaefolia* the leaves are 1-$1\frac{1}{2}$ (rarely $2\frac{1}{2}$ in.) and the petioles $\frac{1}{4}$-$\frac{3}{4}$ in. long, and the flowers in the axils of large leaves or in lax leafy slender spikes or racemes. *L. polyantha* is said to occur in Northern India from the Panjab and N. Scinde to the Concan, Behar and Dacca, ascending the Himalaya

to 6,000 feet, whilst *L. urticaefolia* is stated to be met with ascending the Himalaya to 6,000 feet and its area extending from Jammu to the Nilgiris and into Afganistan and Burma.

According to Sir David Prain's "Bengal Plants" *L. polyantha* and *L. urticaefolia* are distinguished as follows :— *L. polyantha* : leaves usually very shortly petioled, always under $\frac{3}{4}$ in., generally under $\frac{1}{2}$ in. long ; flowers sessile. *L. urticaefolia* : leaves long-petioled, blade 1 in. or more long ; flowers pedicelled.

Although Cooke suggests that *L. polyantha* may be a starved form of *L. urticifolia*, a number of distinguishing marks may be extracted from his Flora of Bombay, namely—

L. urticifolia.	*L. polyantha.*
4-20 in. high	12-20 in. high
leaves $\frac{3}{4}$-2 × $\frac{1}{2}$-1$\frac{1}{4}$"	leaves $\frac{1}{2}$" × $\frac{3}{8}$," rarely longer,
leaves crenate-serrate	leaves serrate-dentate,
petioles $\frac{1}{4}$-1"	petioles very short
flowers solitary or binate in axils of large leaves, sometimes running out into axillary or terminal leafy racemes	flowers numero us in densely leafy racemes in the opposite axils of floral leaves which become smaller upwards
pedicels short	pedicels 0-$\frac{1}{20}$"
calyx-lobes triangular oblong subobtuse	calyx-lobes oblong obtuse
style exceeding the stamens	style as long or rather shorter than the stamens
capsule $\frac{1}{6}$" long	capsule $\frac{1}{4}$" long

From this statement it would appear that Cooke relied chiefly on the size of the leaves, the nature of the inflorescence and the length of the pedicels as discriminating characters.

Collett, in the Flora Simlensis, gives the length of the stem of *L. urticifolia* as 4-12 in. and the size of the leaves as about $\frac{1}{2}$-1$\frac{1}{2}$ × $\frac{1}{2}$-1 in.

According to Duthie (Flora of the Upper Gangetic Plain) *L. polyantha* has a stem 12-20 in. in length, leaves about $\frac{1}{2}$ in. in length and ovate or elliptic, flowers arranged in axillary and terminal leafy spikes, calyx-lobes oblong, ovary pubescent; whilst in *L. urticifolia* the stem is 4-10 in. high, the leaves are $\frac{3}{4}$-2 in. long and broadly ovate, the flowers unilateral, shortly pedicelled solitary or in pairs in the axils of large leaves, sometimes forming

axillary or terminal leafy racemes, the calyx-lobes triangular-oblong, and the ovary pubescent round the apex.

Father Blatter and Prof. Hallberg, after a careful and detailed examination of numerous forms of *L. polyantha* and *L. urticifolia* arrive at the conclusion that *L. polyantha* must be reduced to *L. urticifolia* for the following reasons :.

" (*a*) No distinguishing character of value can be found in the various descriptions published.

(*b*) There is an unbroken chain of intermediate forms uniting the two old species. There are even specimens which, in their different parts, exhibit characters of both the old species.

(*c*) The various descriptions of the plants are often contradictory."

The writer of the present paper agrees entirely with points (*a*) and (*c*), and after the detailed description of the 10 forms by Father Blatter and Prof. Hallberg in the paper already referred to he deems it unnecessary to cite numerous additional instances from the Indo-Gangetic Plains, Rajputana, the Himalayas and Burma. No reliance can be placed on the length of the stem and the size of the leaves, nor on the length of the petiole and pedicel. But notwithstanding this fact .it appears to be possible to distinguish two form-circles, which may safely be treated as species and each of which, on the whole, has a well defined geographical distribution.

An examination of the whole of the material in the Herbarium of the Royal Botanic Gardens, Sibpur, of a set of specimens illustrating the ten forms described by Father Blatter and Prof. Hallberg, of numerous fresh specimens collected by myself in Calcutta and its surroundings, in British Sikkim from Darjiling down to the 3,000 feet level below Kurseong, and of a great number of buds and flowers preserved in formalin and kindly supplied by Mr. Cave of the Lloyd Botanic Garden, Dar iling, indicated that in specimens from a certain geographical area the ovary is invariably glabrous in bud and flower, whilst that from other localities is densely hairy, even in advanced buds and young flowers. The following is a list of the localities in which the specimens examined have been collected, together with the name of the Collector.

I. Specimens with the ovary and style-base densely hairy even in young flowers.

Chamba 3,000′ (C. B. Clarke) ; Simla (Gamble) ; Mussoorie (King) ; Tihri-Gharwal Ganges Valley 4—5,000′ (Duthie) ; Kumaon :

Jeolikot 4,000′ (Gill), Nynee Tal (Kurz), Almora 3,500′ (Hooper); Nipal (Hamilton); Sikkim: (King), Riang (King's Coll.), at 2,000′ (Gammie), in and about Darjiling (Cave, P. Brühl), Tung (P. Brühl) Kurseong (P. Brühl), near Tindaria (P. Brühl), above Ratong River (Anderson); Jaintia Hills (G. Mann); Khasia Hills: Cherrapunji (Burkill and Banerjee), Lohit Valley (Alexander), Naga Hills (C. B. Clarke, D. Prain), Chittagong Hill Tracts (Gamble, C. B. Clarke); Burma: along the Salween (Meebold), Pegan (McLelland), Moulmein; Shan States (Coll.); Upper Burma: at Khampti Long (Toppin), Goteik Gorge 2,500′ (Lace); Yunnan (Henry); Chota Nagpur: (Wood), top of Paresnath (Coll. ?); Rajputana: Mount Abu (King, Blatter).

II. Specimens with ovary and style-base quite glabrous in flower.

Bengal: Dacca (C. B. Clarke), "Lower Bengal" (J. D. Hooker and T. T.), Calcutta (P. Brühl); Upper Indo-Gangetic Plain: Allahabad (Prain), Jaunpur (Coll.), Etawah (Coll.), Lucknow (Anderson); Rajputana: Jodhpur (Coll.), Marwar (Brandis), Gwalior (Maries), Mount Abu (Blatter); Banda (Bell); Bombay Presidency: Khandesh (Blatter), Bassein (Blatter); the Concan (Stokes).

To II belong probably most of the forms described by Father Blatter and Prof. Hallberg.

It appears therefore that the form with a hairy ovary and style-base is essentially Himalayan, extending eastwards into Burma and China and occurring, though rarely, at some higher elevations in the Western Peninsula, particularly on the two mountains sacred to the Jains—Mount Abu and Paresnath—which also in other respects are outliers harbouring certain Himalayan species not occurring in intermediate areas; whilst the form in which the ovary and style-base, evidently before fertilisation has taken place, are quite glabrous is an inhabitant of the Indo-Gangetic plain and the more northern parts of the Western Peninsula. The invasion of the mountainous parts of the Western Peninsula by the form with hairy ovary and style-base finds its counterpart in a similar invasion by *Lindenbergia grandiflora*, a typically Himalayan species, which has been found to occur on the Mahendragiri in the District of Ganjam.

A form, to a certain extent intermediate, has been collected by Lt. S. Toppin on the Malakand. In buds the ovary and style-base are glabrous, in younger flowers they become papillose, and in older flowers—probably after fertilisation has taken place—the *apex* of the

ovary develops hairs; the lower parts of the ovary remain glabrous; ovary and style are 1·3 and 5 mm. in length respectively; the upper lip of the corolla, leaves and habit are those of a typical "L. polyantha."

The statement recorded in the Flora of the British India that the ovary of *L. polyantha* "is certainly hairy" is probably based on the suggestion that Royle's species "ascends the Himalayas to 6,000 feet" and perhaps also on the fact that the young capsule of the form with a glabrous ovary develops hairs at least near its tip, usually also along the margins of the valves; but the capsule of "L. urticifolia" is nearly always more densely hairy all over.

It is a matter of considerable interest to ascertain whether other characters are constantly associated with either of the two forms which may be distinguished by the presence or absence of hairs on ovary and style-base. It will be noticed on examining specimens from an extended area that there is observable a difference in the relative length of ovary and style, as shown by the following record of measurements given in millimeters:

OVARY AND STYLE BASE HAIRY.

Ovary.		*Style.*		*Locality.*
1·0	...	1·5	...	Mount Abu.
2	...	2·2	...	" "
1·3	...	3·0	...	" "
1·5	...	2·3	...	Kumaon.
2·2	...	4·8	...	"
2	...	2·5	...	Sikkim.
1·8	...	3·0	...	"
1·9	...	4·0	...	"
1·5	...	3·8	...	"
1·8	...	3·5	...	Naga Hills.
1·8	...	3·0	...	Khasia Hills.
1·8	...	3·5	...	Jaintia Hills.
1·8	...	4·8	..	Chittagong District.

OVARY AND STYLE-BASE GLABROUS.

1·8	...	8·0	...	Jodhpur.
1·6	...	8·0	...	Gwalior.
2·5	...	7·5	...	"Rajputana."
1·5	...	7·0	...	Allahabad.

Ovary.	*Style.*	*Locality.*
1·7	... 8·0	... Calcutta.
1·9	... 6·0	... „
2·0	... 6·0	... „
1·2	.. 6·0	... Lucknow.
1·3	... 7·0	... Dacca.
1·3	... 4·0	... Bombay.
2·0	... 7·0	... Khandesh.
1·2	... 5·5	... „
1·8	... 5·0	... „
2·0	... 8·5	... Bombay Presidency.
1·2	... 7·8	... Bassein, „

From this it would appear that in the form with the hairy ovary the ratio of ovary to style is 1·0 : 2·6 (rarely 2·8) to 1·0 : 1·1, and in the form with a glabrous ovary 1 : about 7 to 1 : about 3. This probably holds good for the areas referred to above; but if we proceed further east, the distinction fairly breaks down, as shown in the case of specimens with a hairy ovary and style-base—

Ovary.	*Style.*	*Locality.*
1·8	... 4·8	Chittagong District.
1·5	... 4·5	Burma.
1·8	... 5·0	Yunnan.
1·8	... 5·0	Hupe.
2·2	... 6·0	

In the Himalayan and Eastern, as well as in the Mt. Abu forms with hairy ovary the latter passes quite gradually into the slender-conical hairy style-base, whilst in the Indo-Gangetic and Western Peninsula forms the transition from ovary into style is rather abrupt or at least less gradual; this distinction is, however, not absolutely general, and forms occur, especially in the Bombay Presidency, in which the transition from the oblong-ovoid ovary into the filiform style is quite gradual.

A difference which is very noticeable in fresh specimens or specimens preserved in dilute formalin, although ascertainable with considerable difficulty or not at all in herbarium specimens, lies in the shape of the upper corolla-lip. In "L. polyantha" the upper lip is trapezoidal-subtriangular, when flattened out about 3·3 mm. across its base and about 1 mm. across the base of the lobes; these

lobes are obliquely trapezoidal and unequally bidentate or unsymmetrically emarginate; the upper lip of "L. urticifolia" is broad-trapezoidal, when spread out about 4·5 mm. across its base and 2·5 mm. across the base of the lobes, which are subsemicircular or obtusely broad-ovate and nearly entire, the nerves in the Sikkim specimens are parallel or slightly diverging, whilst in the Calcutta form they are converging towards the narrow tip. The palate of the Sikkim plant is more distinctly 4-lobed and markedly transverse; in the Bengal form the groove dividing the two convexities of the palate is either very shallow or obsolete, whilst the lateral weals of the palate extend further down.

A character to which the writer desires to draw the attention of Indian field botanists is one mentioned by Roxburgh in his description of *Stemodia ruderalis* (Flora Indica, III. 95), which is certainly = *Lindenbergia polyantha*, Royle, and not = *L. urticaefolia*, Lehmann. Roxburgh says that the inside of the lips is "beautifully marked with small purple dots." The throat and middle part of the lower lip of the Calcutta plant are always as it were besprinkled with numerous minute well-defined purple or purplish-brown dots; the brownish-red patches which occur on the corolla-tube of the Sikkim form are rather diffuse.

Further, in all the Sikkim specimens examined by me the clefts which separate the calyx-lobes from one another are practically of the same depth, whilst the abaxial cleft is deeper than the four others in all the Bengal specimens seen.

In the Calcutta specimens the oblong-ellipsoidal thecae are purplish-red, their walls shrinking after dehiscence and assuming a brownish-black tint, whilst the pollen is salmon- or flesh-coloured; in the Sikkim plant the walls of the subglobular anther-cells are yellow or nearly colourless and the pollen is colourless.

These are minor differences, but they may on further observations in the field prove to be constant.

In the writer's opinion the fundamental difference of the two form-groups lies in the aestivation of the corolla. Whilst in the Himalayan form the lateral lobes of the lower lip of the corolla overlap the middle lobe in bud, in the Bengal form the middle lobe overlaps the lateral lobes. This statement applies to everyone of the numerous fresh buds examined by the writer in and about Calcutta and in British Sikkim, as well as to the buds preserved in formalin and presented to the writer by Mr. Cave; also to isolated herbarium specimens from other areas, such as Banda. That this difference in

the aestivation of the corolla is a matter of prime importance is shown by the fact that similar differences occur in other form-groups of Lindenbergia; the middle lobe of the lower lip overlaps the lateral lobes in *Lindenbergia Hookeri*, *L. abyssinica*, *L. philippinensis*, *L. macrostachya*, *L. sinaica*, whilst the lateral lobes overlap the middle lobe in *L. grandiflora*, and as far as could be made out from badly preserved material, in *L. Griffithii*. Numerous observations on fresh buds of *L. grandiflora* have convinced the writer that, at least as far as the latter species is concerned, the character referred to is absolutely constant, and will therefore probably be found to be so in the other species mentioned above.

The final conclusion which we arrive at is that we have to deal with two well-defined form-groups, one represented by Linnaeus' *Dodartia indica*, Roxburgh's, Vahl's and probably also Banks' and Retz' *Stemodia ruderalis* and Royle's *Lindenbergia polyantha*, and the other including Roxburgh-Don's *Stemodia muraria* and Lehmann's *Lindenbergia urticaefolia*. These two form-groups appear to be sufficiently distinct to have a claim on being considered "good species." There is no harm in permitting them to continue going under the names of *L. polyantha* and *L. urticifolia* respectively; but considering the confusion that has reigned with regard to them in consequence of botanists attributing to differences in the size of leaves, inflorescence, and pedicels a greater value than they deserve, it appears to be advisable to adopt as the adjectival part of the species names the original terms "indica" and "muraria." Of these the Linnaean term "indica" is particularly appropriate, since the *Lindenbergia polyantha* of Royle is pre-eminently an inhabitant of the Indo-Gangetic Plain and the Western Peninsula and therefore "Indian" in the fullest sense of the word. The word "muraria," although appropriate, would apply equally well to either species; but so does "urticifolia" and "ruderalis." The combination "Lindenbergia indica" was already adopted by Vatke. The writer, therefore, proposes that the names *Lindenbergia indica* (Linn.) and *Lindenbergia muraria* (Roxb.) be adopted as the names of the two form-groups. We would then have—

Lindenbergia indica (Linn.).—Middle lobe of lower corolla-lip overlapping the lateral lobes in aestivation; ovary and style base in (the young) flower quite glabrous; upper lip of corolla trapezoidal-subtriangular shortly bifid with the divisions unsymmetrically bidentate;

LINDENBERGIA MURARIA (Roxb.).—Middle lobe of lower corolla-lip overlapped by the lateral lobes in aestivation; ovary and style-base densely hairy; upper lip of corolla broad-trapezoidal, two-lobed, lobes semicircular or obtusely broad-ovate.

Lindenbergia indica has its nearest relatives in the Arabian *L. sinaica*, (Decaisne) Benth., *L. virens*, Vatke, and *L. fruticosa* Benth., the Tropical African *L. scutellarioides*, Ascherson, the Arabian, Abyssinian and Western Indian *L. abyssinica* and probably some other tropical African species. The relationship appears to be very close between *L. indica* and *L. sinaica*, especially the forms of *L. sinaica* growing on rocks near Aden. This form has smaller flowers than the forms further North. Decaisne describes the flowers as "intense violacei." Ascherson, however, in his description of *L. scutellarioides* remarks that *L. scutellarioides* has "probably, like *L. sinaica*, which has been observed by Schweinfurth in the living state, lemon-yellow corollas suffused posteriorly with reddish-brown." Western Indian Botanists, who may have better opportunities of investigating the Flora of Aden and surroundings than we in Bengal, may be able to state something more positive regarding the relationship of *L. indica* to *L. sinaica* and the colour and structure of the flowers of the latter species.

There is probably no vegetative character in *L. indica* which has not its counterpart in *L. muraria* (*L. urticifolia*). It appears to me, however, that these resemblances are more due to edaphic and climatic influences, than to a closeness of relationship. The aestivation as well as the indumentum of the ovary point to a close connection of *L. muraria* with *L. grandiflora* and *L. Griffithii*. The latter, although it has a glabrous ovary, has a hairy style.

A third form-circle gravitates round *L. philippinensis* and includes *L. siamensis*, Teijsm. &c. Binn., *L. Melvillei*, Moore, and, occupying a somewhat separate position, *L. macrostachya*, Benth. This form-circle will be the subject of a separate note.

As pointed out by Father Blatter and Professor Hallberg and already endorsed by the writer, the forms previously distinguished as *L. urticaefolia* and *L. polyantha* vary greatly in respect to the size and consistency of their leaves, the height of their main stem and the number of flowers in their inflorescence. This is to a great extent due to their being pre-eminently "wall flowers," a preference also shown by other species of Lindenbergia, such as *L. grandifolia*, which is very commonly found growing against walls and rock faces,

or *L. philippinensis* which is stated by E. D. Merrill to grow from the crevices of city walls and by Hance to be met with on walls in Canton, whilst, as noted by Teijsman, *L. siamensis* grows on the walls of pagodas in Bankok. I have met specimens of *L. indica* on a brick-gravel garden path, the gravel being mixed with small quantities of old plaster and ordinary soil. Both *L. indica* and *L. muraria* are also found growing on rocks in chinks and on ledges, further, on road banks, more rarely on cultivated ground or in sandy soil. Species growing in such conditions may naturally be expected to show considerable variations in their vegetative characters. In the following remarks it is assumed that the discriminative characters of *L. indica* (= *L. polyantha*, Royle) and *L. muraria* (= *L. urticaefolia*, Lehm.) are those based on the very constant floral characters referred to above.

The roots of specimens growing in the crevices of walls are often rather long compared with the length of the main stem, especially in the case of individuals which have evidently found considerable difficulty in obtaining the necessary amount of water and soluble mineral matter nearer the surface. The following are some measurements :—root 16 cm., stem 4 cm.; root 10 cm., stem 16 cm.; root 10 cm., stem 8 cm.; on the other hand the root may be more or less fibrous and less than half the length of the stem. Regarding the length of the stem of *L. indica* the writer has come across a specimen the stem of which was only 15 mm. in height, its root measuring 7 mm. in length. In the case of *L. muraria* the length of stem varies between a little more than 1 cm. and rather more than 40 cm.

Both *L. indica* and *L. muraria* are essentially annuals, most of the plants dying and drying up a month or two after the cessation of the monsoon rains. Here and there the basal lateral branches succeed in continuing a precarious existence long after the main stem and most of the other branches have shed their seeds and died.

The size and consistency of the leaves depends probably entirely on the conditions as to soil, moisture and insolation in which the individual plants grow. The following instance will illustrate this statement. The specimens referred to grew within a few hundred feet from each other; their flowers were absolutely similar in every respect except that the parts of the flowers of the shade form were of somewhat larger dimensions. One set grew on the top of an old brick wall in humus gathered by a dense moss growth in the shade of trees; they flowered in August and the beginning of September. These specimens were 7 to 25 cm. in height; the stem was densely, the leaves

sparsely, pubescent; the blade was thin-membranous, coarsely crenate-serrate, of smaller individuals 15 to 24 mm. in length and 8 to 10 mm. in breadth, of more vigorous ones up to 7 cm. in length and about half as broad; the petiole was about two-fifth the length of the blade. The flowers were strictly axillary. In specimens collected on the same wall, but growing from the chinks on the side of the wall and fully exposed to the rays of the sun, the stems were more woody at their base, the leaves were smaller and harder, and the inflorescence was mostly more decidedly of the nature of a raceme. The pedicels of typical specimens of "L. polyantha" vary between 0·1 and 1 mm., in shade plants they may reach 2 mm. An interesting development of the stem was observed in specimens growing on one of the garden walls in one of the streets of Calcutta. The seed had evidently germinated in a crevice on top of the wall, but the growing stem had found itself emerging into the soft humus harbouring a dense growth of moss. The stem had there taken a horizontal course, in one case for about 20 cm., sending out numerous rootlets and producing densely crowded small leaves a few millimeters in diameter, to emerge finally into broad day-light producing numerous lateral ascending branches. There are commonly only two flowers fully open at a time, the flowering continuing for a long period. The flowers turn towards that part of the sky from which the plant receives the maximum amount of light, which causes the inflorescence to be apparently one-sided.

The results of cultivation experiments with both *L. indica* and *L. muraria* as well as those of a histological investigation will be communicated at a later date.

As a great amount of work will yet fall to the share of Indian Field Botanists, I wish to draw their attention to the desirability of carefully noting details in the coloration of the flowers, of preserving specimens of flowers and buds which have only been dried but not pressed and placing them in envelopes on the sheet to which the pressed specimens have been fixed and to preserve, as much as that is possible, flowers and buds in formalin, especially those of Gamopetalae. If this is done systematically, much time, labour and annoyance will be saved to botanists working with dried specimens in Herbaria.

It is the writer's conviction, based on many years' experience, that in the case of species growing over an extensive area, extending for instance from Afganistan along the whole of the Himalaya right into Burma and China, minute floral characters, even small details in the

coloration of the corolla, are often a surer guide in tracing actual relationships than diagnostic characters based on vegetative organs.

The writer is under great obligations to Major A. T. Gage, Superintendent of the Royal Botanic Gardens, Sibpur, and Director of the Botanical Survey of India, and to Mr. C. C. Calder, Curator of the herbarium of the Sibpur Botanical Gardens, for permission to examine in detail the herbarium material of the Royal Botanic Gardens, to Mr. Cave, Superintendant of the Lloyd Botanical Garden, Darjeeling, for presenting the write with a large gathering of flowers and buds, preserved in formalin, of *Lindenbergia muraria* (*L. urticifolia*), and to Father C. Blatter, S.J., for a suite of specimens of the forms of *Lindenbergia indica* (*L. polyantha*) and *L. muraria* described by him and Prof. F. Hallberg.

Botanical Department,
University College of Science,
Baliganj,
The 1st December 1919.

Commentationes Mycologicae

5. *Vermicularia Jatrophae*, Speg.

On *Jatropha integerrima*.

BY

S. N. Bal.

Just after the rains of last year, when collecting fungi in our College compound, it was noticed that the leaves of the beautiful flowering plant, *Jatropha integerrima*, were very badly spotted, and in fact the few plants that were growing, had a stunted appearance. The leaves, thus infected, were collected and examined.

External Characters:—The attack was observed at all parts of the leaves from the margin to the midrib. At some places, the spots were small in size about 1-1·5 cm. in diameter and in some parts the fungal attack had had the effect of the leaves being pierced by holes. The fungus first appears in the form of black dots on the upper surface of the leaves of the host plant and later on, the black dots become encircled by rings of black pycnidia.

The genus *Vermicularia*, as is well known, is characterised by having superficial, erumpent, globose-clavate, carbonaceous, black pycnidia which are beset with long, stiff, septate, dark coloured setae; the conidia borne on the conidiophores being crescent-shaped. In some species, however, the perithecia are imperfect, in others widely open.

On examination of transverse sections through the infested portions of leaves it is found that the perithecium is embedded in the epidermal cells of the leaf tissue. The perithecium is black and measures 60—95μ across. A number of long stiff setae arise from the perithecium. These setae are blackish brown in colour. A number of conidial spores are found at the base of these setae. These spores are almost colourless and are borne on slender conidiophores, which are also almost colourless. There are some granular substance noticed in the spores, which are crescent-shaped and measures 15-20 $\times$ 2·5-3·5μ.

The spores were readily germinated in tap-water. Germination took place after 24 hours and usually 2 germ-tubes are given off from

the two ends of the spores, and fresh infection probably takes place by means of these hyphae penetrating the leaf-tissue.

It was observed that the fungus disappeared during the cold season.

Some other species of the genus, *Vermicularia,* Fries. are known to be causing serious damage to some of our most important cultivated plants.

There are about 130 species of *Vermicularia* described, of which *V. Capsici,* Sydow, is known to cause serious damage to *Capsicum annum,* Linn. The parasite has been dealt with by Butler in detail in his book " Fungi and Disease in Plants." Butler obtained successful inoculations with this fungus on the Cowpea, Val, tomato and brinjal, and the fruit-rot disease of tomato is due to this fungus.

Another species, V. *Circinas,* Berkley, is known to cause serious damage to onions in the United States. The fungus and the disease caused by it, have been described by Stoneman in the Botanical Gazette, 1898, p. 98.

Through the kindness of Dr. E. J. Butler of Pusa the specific determination of the fungus was done by Babu Rohiniranjan Sen of the Mycological Laboratory at Pusa to whom my thanks are due.

Explanation of the Plate.

1. A portion of a leaf of *Jatropha integerrima* infested by *Vermicularia Jatrophae,* Speg.—Natural size.
2. A transverse section through an infested portion of a leaf showing the perithecia, setae and a few conidial spores. × 520.
3. Conidiophores bearing conidia × 440.

BIOLOGICAL LABORATORY,
BOTANICAL DEPARTMENT.
UNIVERSITY COLLEGE OF SCIENCE,
35, Ballyganj Circular Road,
Calcutta the 30th January, 1920.

COMMENTATIONES MYCOLOGICAE.

6. *Phyllosticta Glycosmidis*, SYDOW AND BUTLER, ON *Glycosmis pentaphylla*, CORR.

BY

H. P. CHAUDHURY, M. Sc. (CAL.)

During the last rainy season when collecting fungi in the suburbs of Calcutta, it was noticed that the leaves of *Glycosmis pentaphylla*, Corr., were infested with some leaf-spot fungus. The plants, thus infested were found to be of stunted growth.

Glycosmis pentaphylla, Corr., is of some economic importance, due to the fact that the stems of these shrubs are highly valued as a substitute for tooth brushes, the stems cut in the lengths of about 6 inches, being sold in the market in bundles of 32 pieces at one anna per bundle. They are said to thoroughly disinfect the mouth and effectively cleanse the teeth.

External appearances of the fungus:—Small brownish spots first appear on the upper surface of the leaves which subsequently become dry and brittle. On the spots, thus formed are observable minute perithecia of circular or irregular shape. They are scattered, dotlike and quite prominent measuring 1-3 mm. in diameter. Eventually these spots give rise to holes in the affected parts of the leaves, which finally drop off. As a result of the attack, the plants invariably become stunted.

On examination, of transverse sections through infested portions of leaves, it is seen that the pycnidia are epiphyllous and globular measuring 60-95 in diameter. The pycnidia are full of spores which are oblong and hyaline measuring $5\text{-}8 \times 1\text{-}2\mu$.

The genus *Phyllosticta* is arbitrarily limited to those species in which the spores are less than 15μ lengthwise, larger spored forms having been placed in the genus *Macrophoma*.

Phyllosticta Glycosmidis Syd. et-Butl., as found on *Glycosmis pentaphylla* in Assam, has been described in "Annales Mycologici" Vol. XIV, no 3/4 196 p. 177 by Sydow and Butler.

Several species of the genus *Phyllosticta* are known to cause serious damage to plants, several of which are of economic importance. There are 43 species of Phyllosticta reported from India by Sydow and Butler in "Annales Mycologici," Vol. XIV, 1916.

Phyllosticta solitaria Ell. and Ev., is known to cause serious damage to apples in the United States of America and the details about the attack and remedial measures, have been worked out by Scott and Rorer and published as a Bulletin (Bull. no. 144, United States Bureau of Plant Industry, Washington D. C.). The remedial measures, adopted in U. S. A. in case of "apple blotch" caused by *Phyllosticta solitaria*, Ell. and Ev., are sprayings with Bordeaux mixture.

My thanks are due to Babu Rohini Ranjan Sen of the Pusa Mycological Laboratory for the specific determination of the fungus.

Explanation of the Plate.

A. An infested leaf of *Glycosmis pentaphylla*, Corr.—natural size.

a. A leaf spot caused by *Phyllosticta Glycosmidis*, Syd. et Butl. ×20.

B. Perithecia of *Phyllosticta Glycosmidis*, Syd. et Butl. ×100.

C. A transverse section through an infested portion of a leaf showing the pycnidia and the spores therein. ×110.

D. A pycnidia of *Phyllosticta Glycosmidis*. ×440.

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Calcutta, the 30th January, 1920.

Commentationes Mycologicæ.

7. A short study of *Plicaria repanda* (Wahl.) Rehm. on *Borassus flabellifer*, Linn.

BY

S. N. Bal and H. P. Chaudhury.

During the month of August, 1919, a beautiful saucer-shaped fungus was noticed growing in abundance in the grounds of the Biological Laboratory, Balliganj, under the shade of a large Croton bush on a heap of ripe fruits of *Borassus flabellifer*. The fungus was ascertained to belong to the group of genera formerly included in the genus, *Peziza*.

As is well-known, the genus *Peziza* has been divided by Saccardo and Rehm into several genera, considered by Lindau as sub-genera, of which *Plicaria* and *Tazzetta* are characterised, as pointed out by Rehm, by having their asci turned blue by iodine. The apotheccia at maturity are typically disc or saucer-shaped and they vary greatly in their size.

The cups of the present fungus vary in width at different stages of development from 4 mm. to 10-11 cm. when they are opened and thus forming a plate. They are almost sessile, irregular, smooth and pale amber-coloured. The margin of the cups is slightly inflexed and the whole plant is rather fleshy and at the same time brittle.

On examination of sections cut through the apothecium, a layer of asci with paraphyses is distinctly seen. The asci are cylindrical and 8-spored measuring $130\text{-}145 \times 8\text{-}11\mu$. They are almost colourless and slightly narrowed at the base, and are turned blue with iodine. The spores are unicellular, elliptical and slightly granular having a distinct cell-wall and, when mature, they measure $8\text{-}11 \times 5\text{-}7\mu$. The spores generally emerge from the ascus by bursting through the middle of the ascus. The paraphyses are filiform and transversely septed terminating in spore-like bodies varying in length.

The fungus is a saprophyte, and its life is generally limited to about 8 or 9 days but under favourable conditions it may live for 12 to 14 days. In fact, some of them were kept in the Laboratory under suitable conditions for the purpose of studying the growth of the fungus. Measurements were taken every day. It is to be noted that after the 6th day the cups were quite open and the upper portion

of the fungus took the shape of a saucer. The following is a record of one of the sets of measurements :—

1st day	.	diameter of upper surface of the cup		4 mm.
2nd „	.	Do.	do.	6 „
3rd „	.	Do.	do.	1·5 cm.
4th „	.	Do.	do.	3·5 „
5th „	.	Do.	do.	4·5 „
6th „	.	diameter of the upper surface of the plate		5·5 „
7th „	.	Do.	do.	6·5 „
8th „	.	Do.	do.	7·5 „
9th „	.	Do.	do.	9·0 „
10th „	.	Do.	do.	10·0 „

Some idea of the various stages of development of the plant can be had by consulting the plate attached to this paper.

During the observation of the gradual development of the fungus it was observed that on the 10th day it was quite flat, on the 11th day it began to dry and shrivel and on the 12th day it had the appearance of a dried up crumpled substance.

As has already been noted, iodine turns the walls of asci blue and therefore may be supposed to consist of some compound closely related to starch. It is also possible that the fungus has a certain food value and this question will be further investigated.

Through the kindness of Dr. E. J. Butler of Pusa, the specific determination of the fungus was made by Babu Rohini Ranjan Sen of the Mycological Laboratory at Pusa, to whom our thanks are due.

Explanation of the Plate.

1, 2, 3, 4, 5, 6 and 7, *Plicaria repanda* showing the different stages of development—natural size.

A. Asci and paraphyses × 440.

B. Hyphae × 440.

C. A microphotograph of a section thorough the apothecium of *Plicaria repanda*, (Wahl) Rehm. × 250.

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